PARALLEL WEIGHT 2 POINTS on HILBERT MODULAR

Forum of Mathematics, Sigma (2019), Vol. 7, e27, 36 pages
doi:10.1017/fms.2019.23

PARALLEL WEIGHT 2 POINTS ON HILBERT MODULAR EIGENVARIETIES AND THE PARITY CONJECTURE

CHRISTIAN JOHANSSON[1] and JAMES NEWTON[2]

[1] Department of Mathematical Sciences, Chalmers University of Technology and The University of Gothenburg, SE-412 96 Gothenburg, Sweden;
email: chrjohv@chalmers.se

[2] Department of Mathematics, King's College London, Strand, London WC2R 2LS, UK;
email: j.newton@kcl.ac.uk

Received 31 October 2019; accepted 17 June 2019

Abstract

Let F be a totally real field and let p be an odd prime which is totally split in F. We define and study one-dimensional 'partial' eigenvarieties interpolating Hilbert modular forms over F with weight varying only at a single place v above p. For these eigenvarieties, we show that methods developed by Liu, Wan and Xiao apply and deduce that, over a boundary annulus in weight space of sufficiently small radius, the partial eigenvarieties decompose as a disjoint union of components which are finite over weight space. We apply this result to prove the parity version of the Bloch–Kato conjecture for finite slope Hilbert modular forms with trivial central character (with a technical assumption if $[F:\mathbb{Q}]$ is odd), by reducing to the case of parallel weight 2. As another consequence of our results on partial eigenvarieties, we show, still under the assumption that p is totally split in F, that the 'full' (dimension $1+[F:\mathbb{Q}]$) cuspidal Hilbert modular eigenvariety has the property that many (all, if $[F:\mathbb{Q}]$ is even) irreducible components contain a classical point with noncritical slopes and parallel weight 2 (with some character at p whose conductor can be explicitly bounded), or any other algebraic weight.

2010 Mathematics Subject Classification: 11F33 (primary); 11G40 (secondary)

1. Introduction

1.1. Eigenvarieties near the boundary of weight space. In recent work, Liu, Wan and Xiao [**LWX17**] have shown that, over a boundary annulus in weight space of sufficiently small radius, the eigencurve for a definite quaternion algebra over $\mathbb{Q}$ decomposes as a disjoint union of components which are finite over weight space. On each component, the slope (that is, the p-adic valuation of the U_p-eigenvalue) varies linearly with the weight, and in particular the slope tends to zero as the weight approaches the boundary of the weight space. A notable consequence of this result is that every irreducible component of the eigencurve contains a classical weight 2 point (with p-part of the Nebentypus character of large conductor, so that the corresponding weight character is close to the boundary of weight space).

The purpose of this note is to extend the methods of [**LWX17**] to establish a similar result for certain one-dimensional eigenvarieties interpolating Hilbert modular forms (Proposition 2.7.1). More precisely, we assume that the (odd) prime p splits completely in a totally real field F, and consider 'partial eigenvarieties' whose classical points are automorphic forms for totally definite quaternion algebras over F whose weights are only allowed to vary at a single place $v|p$. One can also consider the slightly more general situation where the assumption is only that the place v is split.

We give two applications of this result. The first application, following methods and results of Nekovář (for example, [**Nek07**]) and Pottharst–Xiao [**PX**], is to establish new cases of the parity part of the Bloch–Kato conjecture for the Galois representations associated to Hilbert modular forms (see the next subsection for a precise statement). The idea of the proof is that, using the results of [**PX**], we can reduce the parity conjecture for a Hilbert modular newform g of general (even) weight to the parity conjecture for parallel weight two Hilbert modular forms, by moving in a p-adic family connecting g to a parallel weight two form. This parallel weight two form will have local factors at places dividing p given by ramified principal series representations (moving to the boundary of weight space corresponds to increasing the conductor of the ratio of the characters defining this principal series representation). Using our results on the partial eigenvarieties, we carry out this procedure in $d = [F : \mathbb{Q}]$ steps, where each step moves one of the weights to two. An additional difficulty when $F \neq \mathbb{Q}$ is to ensure that at each step we move to a point which is noncritical at all the places $v'|p$, so that the global triangulation over the family provided by the results of [**KPX14**] specializes to a triangulation at all of these points. At the fixed $v' = v$, this follows automatically from the construction. At $v' \neq v$ we show, roughly speaking, that the locus of points on the partial eigenvariety which are critical at v' forms a union of connected components, so one cannot move from noncritical to critical points.

The second application (see Section 4) is to establish that every irreducible component of the 'full' eigenvariety (with dimension $[F : \mathbb{Q}] + 1$) for a totally definite quaternion algebra over F contains a classical point with noncritical slopes and parallel weight 2, or any other algebraic weight.

1.2. The Bloch–Kato conjecture. Let g be a normalized cuspidal Hilbert modular newform of weight $(k_1, k_2, \ldots, k_d, w = 2)$ and level $\Gamma_0(\mathfrak{n})$, over a totally real number field F, with each k_i even. We suppose moreover that the automorphic representation associated to g has trivial central character. Here we use the notation and conventions of [Sai09, Section 1] for the weights of Hilbert modular forms, and in particular $w = 2$ corresponds to the central character of the associated automorphic representation having trivial algebraic part. Note that in the body of the text this will correspond to taking $w = 0$ in the weights we define for totally definite quaternion algebras.

Let E be the number field generated by the Hecke eigenvalues $t_v(g)$. For any finite place λ of E, with residue characteristic p, denote by $V_{g,\lambda}$ the two-dimensional (totally odd, absolutely irreducible) E_λ-representation associated to g which satisfies

$$\det(X - \mathrm{Frob}_v|_{V_{g,\lambda}}) = X^2 - t_v(g)X + q_v$$

for all $v \nmid \mathfrak{n}p$ (Frob_v denotes arithmetic Frobenius).

Note that we have $V_{g,\lambda} \cong V^*_{g,\lambda}(1)$. The conjectures of Bloch and Kato predict the following formula relating the dimension of the Bloch–Kato Selmer group of $V_{g,\lambda}$ to the central order of vanishing of an L-function:

CONJECTURE 1.2.1.

$$\dim_{E_\lambda} H^1_f(F, V_{g,\lambda}) = \mathrm{ord}_{s=0} L(V_{g,\lambda}, s) \text{ (or, equivalently, } = \mathrm{ord}_{s=0} L(g, s)) .$$

In this conjecture, the L-functions are normalized so that the local factors (at good places) are

$$L_v(V_{g,\lambda}, s) = L_v(g, s) = \left(1 - t_v q_v^{-s-1} + q_v^{-2s-1}\right)^{-1} .$$

We refer to the following as the 'parity conjecture for $V_{g,\lambda}$':

CONJECTURE 1.2.2.

$$\dim_{E_\lambda} H^1_f(F, V_{g,\lambda}) \equiv \mathrm{ord}_{s=0} L(V_{g,\lambda}, s) \pmod 2.$$

Our main result towards the parity conjecture is the following, which is proved in Section 3:

THEOREM 1.2.3. *Let p be an odd prime and let F be a totally real number field in which p splits completely. Let g be a cuspidal Hilbert modular newform for F with weight $(k_1, k_2, \ldots, k_d, 2)$ (each k_i even) such that the associated automorphic representation π has trivial central character. Let E be the number field generated by the Hecke eigenvalues of g and let λ be a finite place of E with residue characteristic p.*

Suppose that for every place $v|p$, π_v is not supercuspidal (in other words, g has finite slope, up to twist, for every $v|p$). If $[F:\mathbb{Q}]$ is odd suppose moreover that there is a finite place $v_0 \nmid p$ of F such that π_{v_0} is not principal series. Then the parity conjecture holds for $V_{g,\lambda}$.

REMARK 1.2.4.

(1) For fixed g there is a positive density set of primes p satisfying the assumptions of the above theorem. So combining our theorem with the conjectural 'independence of p' for the (parity of the) rank of the Selmer group $H^1_f(F, V_{g,\lambda})$ would imply the parity conjecture for the entire compatible family of Galois representations associated to g (with the assumption on the existence of a suitable v_0 if $[F:\mathbb{Q}]$ is odd).

(2) When $k_i = 2$ for all i, the above theorem is already contained in [PX]. Moreover, stronger results (without the finite slope hypothesis) are given by [Nek13, Theorem 1.4] and [Nek18, Theorem C]. As far as the authors are aware, the only prior results for higher weights are for p-ordinary forms (for example, [Nek06, Theorem 12.2.3]), except when $F = \mathbb{Q}$ in which case the above theorem follows from the results of [PX] and [LWX17] (cf. [LWX17, Remark 1.10]), and there is also a result of Nekovář which applies under a mild technical hypothesis [Nek13, Theorem B].

(3) When $[F:\mathbb{Q}]$ is odd we require the existence of the place v_0 in the above theorem in order to switch to a totally definite quaternion algebra.

2. A halo for the partial eigenvariety

2.1. Notation. We fix an odd prime number p, a totally real number field F with $[F:\mathbb{Q}] = d > 1$, and we assume that p splits completely in F. The assumption that p is odd is not essential for this section, but it appears in the results of [PX] and [Nek18] which we will apply later, so we have excluded the case $p = 2$ throughout this paper for simplicity. We also fix a totally definite quaternion algebra D over F, with discriminant δ_D, and assume that p is coprime to δ_D. We fix a maximal order $\mathcal{O}_D \subset D$ and an isomorphism $\mathcal{O}_D \otimes_{\mathcal{O}_F} \widehat{\mathcal{O}}_F^{\delta_D} \cong M_2(\widehat{\mathcal{O}}_F^{\delta_D}) = \prod_{v \nmid \delta_D} M_2(\mathcal{O}_{F,v})$. Using this isomorphism, we henceforth identify $\mathcal{O}_{D,p}^\times$ and $\prod_{v|p} \mathrm{GL}_2(\mathcal{O}_{F,v})$. We fix the uniformizer $\varpi_v = p$ of $\mathcal{O}_{F,v}$ for each $v|p$.

For $v|p$, we let $T_{0,v} = \begin{pmatrix} \mathcal{O}_{F,v}^{\times} & 0 \\ 0 & \mathcal{O}_{F,v}^{\times} \end{pmatrix} \subset \mathrm{GL}_2(\mathcal{O}_{F,v})$, let

$$I_{v,n} = \left\{ \begin{pmatrix} a & b \\ \varpi_v^n c & d \end{pmatrix} \mid a, b, c, d \in \mathcal{O}_{F,v} \right\} \cap \mathrm{GL}_2(\mathcal{O}_{F,v}),$$
$$\overline{I}_{v,n} = \left\{ \begin{pmatrix} a & \varpi_v^n b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathcal{O}_{F,v} \right\} \cap \mathrm{GL}_2(\mathcal{O}_{F,v})$$

and

$$I_{v,1,n} = \left\{ \begin{pmatrix} a & b \\ \varpi_v c & d \end{pmatrix} \in I_{v,1} \mid b \equiv 0 \pmod{\varpi_v^n} \right\},$$
$$I'_{v,1,n} = \left\{ \begin{pmatrix} a & \varpi_v^n b \\ \varpi_v c & d \end{pmatrix} \in I_{v,1,n} \mid a \equiv d \equiv 1 \pmod{\varpi_v^{n+1}} \right\}.$$

Set

$$\Sigma_v^+ = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \varpi_v^{\beta} \end{pmatrix} \mid \beta \in \mathbb{Z}_{\geqslant 0} \right\}$$

and

$$\Sigma_v^{cpt} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \varpi_v^{\beta} \end{pmatrix} \mid \beta \in \mathbb{Z}_{>0} \right\}.$$

The groups $I_{v,1,n}$ will be convenient to work with later; see Lemma 2.3.1 and Remark 2.3.2. Fix a place $v|p$ (this will be the place where the weight varies in our families of p-adic automorphic forms), and let $K^v = \prod_{v' \neq v} K_{v'}$ be a compact open subgroup of $(\mathcal{O}_D \otimes_F \mathbb{A}_{F,f}^v)^{\times}$ such that $K = K^v I_{v,1}$ is a *neat* subgroup of $(D \otimes_F \mathbb{A}_{F,f})^{\times}$. More precisely, we assume that $x^{-1} D^{\times} x \cap K \subset \mathcal{O}_F^{\times,+}$ (the totally positive units) for all $x \in (D \otimes_F \mathbb{A}_{F,f})^{\times}$.

We make the following general definitions:

DEFINITION 2.1.1. Let K be a compact open subgroup of $(D \otimes_F \mathbb{A}_{F,f})^{\times}$ and let N be a monoid with $K \subset N \subset (D \otimes_F \mathbb{A}_{F,f})^{\times}$. Suppose M is a left $R[N]$-module (for some commutative coefficient ring R).

(1) If $f : D^{\times} \backslash (D \otimes_F \mathbb{A}_{F,f})^{\times} \to M$ is a function and $\gamma \in N$ we define a function ${}_{\gamma}|f$ by ${}_{\gamma}|f(g) = \gamma f(g\gamma)$.

(2) We define

$$H^0(K, M) = \{ f : D^{\times} \backslash (D \otimes_F \mathbb{A}_{F,f})^{\times} \to M \mid {}_{u}|f = f \text{ for all } u \in K \}.$$

(3) If $K' \subset K$ is a compact open subgroup then we can define a double coset operator
$$[KgK'] : H^0(K', M) \to H^0(K, M)$$
for any $g \in N$ by decomposing the double coset $KgK' = \coprod_i g_i K'$ and defining
$$[KgK']f = \sum_i {}_{g_i}|f.$$

The following lemma is easily checked.

LEMMA 2.1.2. *Suppose K is a neat compact open subgroup of $(D \otimes_F \mathbb{A}_{F,f})^\times$. Let $g_1, \dots, g_t$ be double coset representatives for $D^\times \backslash (D \otimes_F \mathbb{A}_{F,f})^\times / K$, and let M be an $R[K]$-module on which $Z(K) := \mathcal{O}_F^\times \cap K = \mathcal{O}_F^{\times,+} \cap K$ acts trivially (the equality follows from the neatness assumption).*

Then the R-module map
$$f \mapsto (f(g_1), \dots, f(g_t))$$
gives an isomorphism $H^0(K, M) \cong M^{\oplus t}$.

Now suppose we have an integer w and a tuple of integers
$$k^v = (k_{v'})_{v'|p, v' \neq v}$$
such that $k_{v'} \equiv w \pmod 2$ and $k_{v'} \geqslant 2$ for all v'.

For each v' we define a finite free $\mathbb{Z}_p = \mathcal{O}_{F_{v'}}$-module with a left action of $K_{v'}$:
$$\mathcal{L}_{v'}(k_{v'}, w) = \det{}^{(w-k_{v'}+2)/2} \operatorname{Sym}^{k_{v'}-2} \mathcal{O}_{F_{v'}}^2,$$
where $\mathcal{O}_{F_{v'}}^2$ is the standard representation of $\mathrm{GL}_2(\mathcal{O}_{F,v'}) \cong \mathcal{O}_{D,v'}^\times$.

We then define a finite free $\mathbb{Z}_p$-module with a left action of K^v:
$$\mathcal{L}^v(k^v, w) := \bigotimes_{v'|p, v' \neq v} \mathcal{L}_{v'}(k_{v'}, w).$$

Let χ^v be the character of $T_0^v = \prod_{v'|p, v' \neq v} T_{0,v'}$ given by the highest weight of $\mathcal{L}^v(k^v, w)$ (with respect to the upper triangular Borel). Explicitly, this character is
$$\chi^v(t_1, t_2) := \prod_{v'} (t_{1,v'})^{(w+k_{v'}-2)/2} (t_{2,v'})^{(w-k_{v'}+2)/2}.$$

We now recall some terminology from [JN19b]. We let R be a Banach–Tate $\mathbb{Z}_p$-algebra [JN19b, 3.1] with a multiplicative pseudouniformizer ϖ, and let

$\kappa : T_{0,v} \to R^\times$ be a continuous character. Assume that the norm on R is adapted to κ [JN19b, Definition 3.3.2] and that $\chi^v\kappa$ is trivial on $Z(K)$. We call such a κ a *weight*. Associated with a weight κ is a number $r_\kappa \in [1/p, 1)$ measuring how 'analytic' κ is; see [JN19b, Definition 3.3.9] for the precise definition.

For $r \geqslant r_\kappa$ we get a Banach R-module $\mathcal{D}_\kappa^r$ [JN19b, Definition 3.3.9] equipped with a left action of the monoid $\Delta_v = I_{v,1}\Sigma_v^+ I_{v,1}$. This can be thought of as a space of distributions, which acts (see the end of [JN19b, Section 3.3]) on a space of functions $\mathcal{A}_\kappa^{<r}$ on $\begin{pmatrix} 1 & 0 \\ \varpi_v\mathcal{O}_{F,v} & 1\end{pmatrix}$, which we identify with functions in a single variable x on $\varpi_v\mathcal{O}_{F,v}$. $\mathcal{A}_\kappa^{<r}$ carries a right action of Δ_v, with the action of an element $\gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Delta_v$ given by

$$(f\cdot\gamma)(x) = \kappa(a+bx, (\det\gamma)|\det\gamma|_v/(a+bx))f\left(\frac{c+dx}{a+bx}\right).$$

We note in passing that the action pairing $\mathcal{D}_\kappa^r \times \mathcal{A}_\kappa^{<r} \to R$ identifies $\mathcal{A}_\kappa^{<r}$ as the dual of $\mathcal{D}_\kappa^r$ (and not the other way around!). The R-module $H^0(K, \mathcal{L}^v(k^v, w)\otimes_{\mathbb{Z}_p}\mathcal{D}_\kappa^r)$ is the space of p-adic automorphic forms with fixed weights away from v which we will be studying in this section.

We have a natural action of $(D\otimes_F \mathbb{A}_{F,f}^p)^\times \times \Delta_v$ on $\mathcal{L}^v(k^v, w)\otimes_{\mathbb{Z}_p}\mathcal{D}_\kappa^r$, and we get associated double coset operators. If R is a $\mathbb{Q}_p$-algebra, we can extend this to an action of $(D\otimes_F \mathbb{A}_{F,f}^v)^\times \times \Delta_v$. We will especially consider the action on $H^0(K, \mathcal{L}^v(k^v, w)\otimes_{\mathbb{Z}_p}\mathcal{D}_\kappa^r)$ of the Hecke operator $U_v = [K\left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_v\end{smallmatrix}\right)K]$. This Hecke operator is compact because, by [JN19b, Corollary 3.3.10], it factors as a composition

$$\begin{aligned} H^0(K, \mathcal{L}^v(k^v, w)\otimes_{\mathbb{Z}_p}\mathcal{D}_\kappa^r) &\to H^0(K, \mathcal{L}^v(k^v, w)\otimes_{\mathbb{Z}_p}\mathcal{D}_\kappa^{r^{1/p}}) \\ &\hookrightarrow H^0(K, \mathcal{L}^v(k^v, w)\otimes_{\mathbb{Z}_p}\mathcal{D}_\kappa^r), \end{aligned}$$

where the second map is induced by the natural compact inclusion $\mathcal{D}_\kappa^{r^{1/p}} \hookrightarrow \mathcal{D}_\kappa^r$.

When R is a $\mathbb{Q}_p$-algebra we will also use the (noncompact) Hecke operators $U_{v'} = p^{-(w-k_{v'}+2)/2}[K\left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_{v'}\end{smallmatrix}\right)K]$ for the places $v \neq v'|p$. We have normalized the operators $U_{v'}$ so that they are consistent with U_v. In particular, when R is a field extension of $\mathbb{Q}_p$, the nonzero eigenvalues for $U_{v'}$ will have p-adic valuation between 0 and $k_{v'}-1$.

2.2. Locally algebraic weights. Suppose $\kappa : T_{0,v} \to E^\times$ is a weight, where E is a finite extension of $\mathbb{Q}_p$, and for some positive integer k with $k \equiv w \pmod 2$

the restriction of κ to an open subgroup $(1+\varpi_v^n\mathcal{O}_{F_v})^2$ of $T_{0,v}$ coincides with the character

$$\chi_k(t_1,t_2)=(t_1)^{(w+k-2)/2}(t_2)^{(w-k+2)/2}.$$

Then, denoting the finite order character κ/χ_k by ϵ, we say that κ is *locally algebraic of weight* (k,w) *and character* ϵ. If we make the standard identification of the E-dual (right) representation $\mathcal{L}_v(k,w)^\vee$ with homogeneous degree k polynomials in two variables (X,Y) then evaluation at $(1,x)$ gives an injective $I'_{v,1,n-1}$-equivariant map

$$\mathcal{L}_v(k,w)^\vee\to\mathcal{A}_\kappa^{<r}.$$

The action of $\left(\begin{smallmatrix}1&0\\0&\varpi_v\end{smallmatrix}\right)$ on the right hand side induces the action of $p^{-(w-k+2)/2}\left(\begin{smallmatrix}1&0\\0&\varpi_v\end{smallmatrix}\right)$ on the left hand side. The action of $\mathcal{D}_\kappa^r$ on $\mathcal{L}_v(k,w)^\vee$ then gives us a surjective $I'_{v,1,n-1}$-equivariant map

$$\mathcal{D}_\kappa^r\to\mathcal{L}_v(k,w)\otimes_{\mathbb{Z}_p}E.$$

Moreover, if we let $\mathcal{L}_v(k,w,\epsilon)$ denote the $I_{v,1,n-1}$-representation obtained from $\mathcal{L}_v(k,w)\otimes_{\mathbb{Z}_p}E$ by twisting the action by ϵ then we obtain a surjective $I_{v,1,n-1}$-equivariant map

$$\mathcal{D}_\kappa^r\to\mathcal{L}_v(k,w,\epsilon).$$

2.3. Comparing level $I_{v,1}$ and $I_{v,1,n-1}$. We retain the set-up of the previous subsection, so κ is a locally algebraic weight. A version of the following lemma is commonly used in Hida theory (see for example [Ger19, Lemma 2.5.2]):

LEMMA 2.3.1.

(1) *For $n\geqslant 2$, the endomorphism U_v of $H^0(K^vI_{v,1,n-1},\mathcal{L}^v(k^v,w)\otimes_{\mathbb{Z}_p}\mathcal{D}_\kappa^r)$ factors through the natural inclusion*

$$H^0(K^vI_{v,1,n-2},\mathcal{L}^v(k^v,w)\otimes_{\mathbb{Z}_p}\mathcal{D}_\kappa^r)\hookrightarrow H^0(K^vI_{v,1,n-1},\mathcal{L}^v(k^v,w)\otimes_{\mathbb{Z}_p}\mathcal{D}_\kappa^r).$$

(2) *The natural inclusion*

$$\iota:H^0(K^vI_{v,1},\mathcal{L}^v(k^v,w)\otimes_{\mathbb{Z}_p}\mathcal{D}_\kappa^r)\hookrightarrow H^0(K^vI_{v,1,n-1},\mathcal{L}^v(k^v,w)\otimes_{\mathbb{Z}_p}\mathcal{D}_\kappa^r)$$

is U_v-equivariant and if $h\in\mathbb{Q}_{\geqslant 0}$ it induces an isomorphism between U_v-slope $\leqslant h$ subspaces:

$$\iota:H^0(K^vI_{v,1},\mathcal{L}^v(k^v,w)\otimes_{\mathbb{Z}_p}\mathcal{D}_\kappa^r)^{\leqslant h}\cong H^0(K^vI_{v,1,n-1},\mathcal{L}^v(k^v,w)\otimes_{\mathbb{Z}_p}\mathcal{D}_\kappa^r)^{\leqslant h}.$$

Proof. First we note that (for $n \geqslant 1$) the action of U_v on $H^0(K^v I_{v,1,n-1}, \mathcal{L}^v(k^v, w) \otimes_{\mathbb{Z}_p} \mathcal{D}_\kappa^r)$ is given by

$$f \mapsto \sum_{i=0}^{p-1} \left(\begin{smallmatrix} 1 & 0 \\ \varpi_v i & \varpi_v \end{smallmatrix}\right) | f.$$

The U_v-equivariance in the second statement follows immediately from this. The rest of the second statement follows from the first, since U_v acts invertibly on the slope $\leqslant h$ subspace.

It follows from a simple computation that the double coset operator

$$[I_{v,1,n-1} \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_v \end{smallmatrix}\right) I_{v,1,n-1}]$$

is equal to $[I_{v,1,n-2} \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_v \end{smallmatrix}\right) I_{v,1,n-1}]$, and the first part follows immediately from this. □

REMARK 2.3.2. The clean statement of Lemma 2.3.1(2) is the reason why we have chosen to work with the nonstandard groups $I_{v,1,n-1}$ instead of the $I_{v,n}$. The reader may compare this with [**Buz07**, Lemma 11.1]; there one works, roughly speaking, with the $I_{v,n}$ but the map relating forms of level $K^v I_{v,1}$ to forms of level $K^v I_{v,n}$ is not the natural inclusion.

Having done all this, composing ι with the map induced by $\mathcal{D}_\kappa^r \to \mathcal{L}_v(k, w, \epsilon)$, we obtain a Hecke-equivariant map

$$\pi : H^0(K^v I_{v,1}, \mathcal{L}^v(k^v, w) \otimes_{\mathbb{Z}_p} \mathcal{D}_\kappa^r) \to H^0(K^v I_{v,1,n-1}, \mathcal{L}^v(k^v, w) \otimes_{\mathbb{Z}_p} \mathcal{L}_v(k, w, \epsilon)), \tag{2.3.1}$$

where the action of U_v on the target is the $\star$-*action* defined by multiplying the standard action of U_v by $p^{-(w-k+2)/2}$.

PROPOSITION 2.3.3. *Let $h \in \mathbb{Q}_{\geqslant 0}$ with $h < k - 1$. The map π induces an isomorphism between U_v-slope $\leqslant h$ subspaces.*

Proof. By Lemma 2.3.1 it suffices to show that the map

$$H^0(K^v I_{v,1,n-1}, \mathcal{L}^v(k^v, w) \otimes_{\mathbb{Z}_p} \mathcal{D}_\kappa^r) \to H^0(K^v I_{v,1,n-1}, \mathcal{L}^v(k^v, w) \otimes_{\mathbb{Z}_p} \mathcal{L}_v(k, w, \epsilon))$$

induces an isomorphism between U_v-slope $\leqslant h$ subspaces, and this can be proved as in [**Han17**, Theorem 3.2.5]. □

2.4. The 'Atkin–Lehner trick'. In this section we establish a result analogous to [**LWX17**, Proposition 3.22]. Let $n \geqslant 2$ be an integer and suppose

$$K_{v'} \subset \begin{pmatrix} 1+\varpi_{v'}^{n}\mathcal{O}_{F,v'} & \mathcal{O}_{F,v'} \\ \varpi_{v'}^{n}\mathcal{O}_{F,v'} & 1+\varpi_{v'}^{n}\mathcal{O}_{F,v'} \end{pmatrix}$$

for each $v'|p$, $v' \neq v$. Note that, combined with our neatness assumption, this implies that $Z(K)$ is contained in $1+p^{n}\mathcal{O}_{F}$.

Fix an integer $k \geqslant 2$ with the same parity as w, together with a finite order character $\epsilon = (\epsilon_1, \epsilon_2) : T_{0,v} \to E^{\times}$, where ϵ_1 and ϵ_2 are characters of $(\mathcal{O}_{F,v}/\varpi_v^n)^{\times}$ and ϵ_2/ϵ_1 has conductor (ϖ_v^n).

Let $\epsilon_{\mathbb{Q}}$ be the finite order Hecke character of $\mathbb{Q}^{\times}\backslash\mathbb{A}_{\mathbb{Q}}^{\times}$ associated to the Dirichlet character

$$(\mathbb{Z}/p^n\mathbb{Z})^{\times} \cong (\mathcal{O}_{F,v}/\varpi_v^n)^{\times} \overset{\epsilon_1\epsilon_2}{\to} E^{\times}.$$

We now consider the space of classical automorphic forms $S(k, w, \epsilon) := H^0(K^v I_{v,1,n-1}, \mathcal{L}^v(k^v, w) \otimes \mathcal{L}(k, w, \epsilon))$. By Lemma 2.1.2, the dimension of this space is equal to $(k-1)p^{n-1}t$ (as we noted above, $Z(K) \subset 1+p^n\mathcal{O}_F$, so it acts trivially on the coefficients), where

$$t = |D^{\times}\backslash(D \otimes_F \mathbb{A}_{F,f})^{\times}/K^v I_{v,1}| \prod_{v'|p, v' \neq v} (k_{v'} - 1).$$

Denote the slopes of U_v appearing in $S(k, w, \epsilon)$ by $\alpha_0(\epsilon), \ldots, \alpha_{(k-1)p^{n-1}t-1}(\epsilon)$ in nondecreasing order. If γ an element of some field, we let $ur(\gamma)$ be the character on $F_v^{\times}$ which is trivial on $\mathcal{O}_{F_v}^{\times}$ and takes uniformizers to γ.

LEMMA 2.4.1. *We have* $\alpha_i(\epsilon) = k - 1 - \alpha_{(k-1)p^{n-1}t-1-i}(\epsilon^{-1})$ *for* $i = 0, \ldots, (k-1)p^{n-1}t - 1$. *In particular, the sum (with multiplicities) of the* U_v *slopes appearing in* $S(k, w, \epsilon) \oplus S(k, w, \epsilon^{-1})$ *is* $(k-1)^2 p^{n-1} t$.

Proof. First we fix an embedding $\iota : E \hookrightarrow \mathbb{C}$. Using this embedding, we regard $\epsilon_{\mathbb{Q}}$ and ϵ as complex valued characters. The space $S(k, w, \epsilon) \otimes_{E,\iota} \mathbb{C}$ can be described in terms of automorphic representations for D. If π contributes to this space, the local factor π_v is a principal series representation of $\mathrm{GL}_2(F_v)$ obtained as the normalized induction of a pair of characters

$$ur(\alpha^{-1}\zeta p^{w})\epsilon_1^{-1} \times ur(\alpha)\epsilon_2^{-1}$$

where ζ is a root of unity. To see this, we first note that π_v has a nonzero subspace on which $I_{v,1,n-1}$ acts via the character ϵ^{-1} (these are the vectors in π_v which contribute to $S(k, w, \epsilon) \otimes_{E,\iota} \mathbb{C}$). So $\pi_v \otimes \epsilon_2 \circ \det$ has a nonzero subspace on which $I_{v,1,n-1}$ acts via $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto (\epsilon_2/\epsilon_1)(a)$. Applying $\begin{pmatrix} 1 & 0 \\ 0 & \varpi_v^{n-1} \end{pmatrix}$ to this subspace, we get a nonzero subspace where the action of $I_{v,n}$ is given by the same formula. Therefore the conductors of both $\pi_v \otimes \epsilon_2 \circ \det$ and its central character ϵ_2/ϵ_1 are (ϖ_v^n). It follows (for example, by [Tem14, Lemma 3.3]) that $\pi_v \otimes \epsilon_2 \circ \det$ is the normalized induction of $\mu_1 \times \mu_2$ with $\mu_1|_{\mathcal{O}_{F,v}^\times} = \epsilon_2/\epsilon_1$ and $\mu_2|_{\mathcal{O}_{F,v}^\times} = 1$. The rest of the claim is deduced from the fact that the central character of π_v is the product of a finite order unramified character and $\epsilon_1^{-1}\epsilon_2^{-1}|\cdot|_v^{-w}$.

We now need to compute the eigenvalue for the standard U_v action on the subspace of π_v where $I_{v,1,n-1}$ acts via $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \epsilon_1(a)^{-1}\epsilon_2(d)^{-1}$. Conjugating by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi \end{smallmatrix}\right)$, we can do the same for the standard U_v action defined with respect to $\overline{I}_{v,n}$, and we get U_v-eigenvalue $\alpha p^{1/2}$. Using the $\star$-action we therefore get U_v-eigenvalue $\alpha p^{(k-1-w)/2}$.

Twisting by $\epsilon_{\mathbb{Q}}$ gives a π' with local factor π'_v the normalized induction of

$$ur(\alpha^{-1}\zeta p^w)\epsilon_2 \times ur(\alpha)\epsilon_1,$$

which contributes to $S(k, w, \epsilon^{-1})$ with $\star$-action U_v-eigenvalue $\alpha^{-1}\zeta p^{(w+k-1)/2}$. Summing the two slopes together we get $(k-1)$, which gives the desired result. □

2.5. The weight space of the partial eigenvariety. We are going to construct a 'partial eigenvariety' out of the spaces $H^0(K, \mathcal{L}^v(k^v, w) \otimes_{\mathbb{Z}_p} \mathcal{D}_\kappa^r)$. The underlying weight space $\mathcal{W}$ is defined by letting

$$\mathcal{W}(A) = \{\kappa \in \mathrm{Hom}_{cts}(T_{0,v}, A^\times) : \chi^v \kappa|_{Z(K)} = 1\}$$

for algebras A of topologically finite type over $\mathbb{Q}_p$. We let Δ denote the torsion subgroup of $\mathcal{O}_{F,v}^\times$.

LEMMA 2.5.1. *We have an isomorphism $\mathcal{W}_{\mathbb{C}_p} \cong \coprod_{\eta,\omega} D_{\mathbb{C}_p}$ where D is the open unit disc and η, ω run over pairs of characters $\eta : \mathcal{O}_v^\times \to \mathbb{C}_p^\times, \omega : \Delta \times \Delta \to \mathbb{C}_p^\times$ such that $\chi^v\eta|_{Z(K)} = 1$ and ω restricted to the diagonal copy of Δ is equal to $\eta|_\Delta$.*

The isomorphism is given by taking κ to $\kappa(\exp(p), \exp(p)^{-1}) - 1$ in the component labelled by $\eta(t_v) = \kappa(t_v, t_v)$ and $\omega(\delta_1, \delta_2) = \kappa(\delta_1, \delta_2)$.

Proof. This follows from the fact that the closure of $Z(K)$ in $T_{0,v}$ is a finite index subgroup of $\mathcal{O}_v^\times$ centrally embedded in $T_{0,v}$. □

REMARK 2.5.2. The condition that $\chi^v\eta|_{Z(K)} = 1$ is equivalent to the condition that $\eta(t_v) = t_v^w\tilde{\eta}(t_v)$ where $\tilde{\eta}$ is a finite order character, trivial on $Z(K)$.

In particular, there are finitely many possibilities for η. Moreover, for each η, if we denote by E_η the finite extension of $\mathbb{Q}_p$ generated by the values of η then the open and closed immersion $\coprod_\omega D_{\mathbb{C}_p} \hookrightarrow \mathcal{W}_{\mathbb{C}_p}$ given by restricting to the components labelled by η is defined over E_η. For each pair η, ω we therefore denote by $\mathcal{W}_{\eta,\omega}$ the corresponding connected component (which is isomorphic to the open unit disc over E_η) of $\mathcal{W}_{E_\eta}$.

If κ is a point of $\mathcal{W}_{\mathbb{C}_p}$ we denote by z_κ the corresponding point of $D_{\mathbb{C}_p}$. If $r \in (0, 1) \cap \mathbb{Q}$, the union of open annuli given by $r < |z_\kappa|_p < 1$ is denoted by $\mathcal{W}^{>r}$.

2.6. Lower bound for the Newton polygon. In this section we establish a result analogous to [LWX17, Theorem 3.16], following the approach of [JN19b, Section 6.2]. We now return to our general situation: R is a Banach–Tate $\mathbb{Z}_p$-algebra with a multiplicative pseudouniformizer ϖ, and $\kappa : T_{0,v} \to R^\times$ is a continuous character such that the norm on R is adapted to κ and $\chi^v\kappa$ is trivial on $Z(K)$.

As in [JN19b, (3.2.1)], for $\alpha \in \mathbb{Z}_{\geqslant 0}$, we define

$$n(r, \varpi, \alpha) = \left\lfloor \frac{\alpha \log_p r}{\log_p |\varpi|} \right\rfloor.$$

LEMMA 2.6.1. *Assume that there is no $x \in R$ with $1 < |x| < |\varpi|^{-1}$. Let*

$$t = |D^\times \backslash (D \otimes_F \mathbb{A}_{F,f})^\times / K^v I_{v,1}| \prod_{v'|p, v' \neq v} (k_{v'} - 1).$$

If we define

$$\lambda(0) = 0, \lambda(i+1) = \lambda(i) + n(r, \varpi, \lfloor i/t \rfloor) - n(r^{1/p}, \varpi, \lfloor i/t \rfloor),$$

then the Fredholm series

$$\det(1 - TU_v | H^0(K^v I_{v,1}, \mathcal{L}^v(k^v, w) \otimes_{\mathbb{Z}_p} \mathcal{D}_\kappa^r)) = \sum_{n \geqslant 0} c_n T^n \in R\{\{T\}\}$$

satisfies $|c_n| \leqslant |\varpi|^{\lambda(n)}$.

Proof. This is essentially [JN19b, Lemma 6.2.1]. Our global set-up is slightly different to that in [JN19b, Lemma 6.2.1], but the proof goes through verbatim. □

We now fix a component $\mathcal{W}_{\eta,\omega}$ of weight space. Let $E := E_\eta \subset \mathbb{C}_p$ be the subfield generated by the image of η (it is generated by a p-power root of unity). We fix a uniformizer $\varpi_E \in E$ and normalize the absolute value $|\cdot|$ on E by $|p| = p^{-1}$. Let $\Lambda = \mathcal{O}_E[[X]]$. We have a universal weight

$$\kappa_{(\eta,\omega)} : T_{0,v} \to \Lambda^\times$$

determined by $\kappa|_{\Delta\times\Delta} = \omega$, $\kappa(t_v, t_v) = \eta(t_v)$ and $\kappa(\exp(p), \exp(p)^{-1}) = 1 + X$.

We give the complete local ring Λ the $\mathfrak{m}_\Lambda = (\varpi_E, X)$-adic topology. Let $\mathfrak{W}_{\eta,\omega} = \mathrm{Spa}(\Lambda, \Lambda)$, denote its analytic locus by $\mathcal{W}^{an}_{\eta,\omega}$ and let $\mathcal{U}_1 \subset \mathcal{W}^{an}_{\eta,\omega}$ be the rational subdomain defined by

$$\mathcal{U}_1 = \{|\varpi_E| \leqslant |X| \neq 0\}.$$

Pulling back $\mathcal{U}_1$ to the rigid analytic open unit disc $\mathcal{W}_{\eta,\omega}$ gives the 'boundary annulus' $|X| \geqslant |\varpi_E|$.

We let $R = \mathcal{O}(\mathcal{U}_1)$. More explicitly, we can describe the elements of R° as formal power series

$$\left\{\sum_{n\in\mathbb{Z}} a_n X^n : a_n \in \mathcal{O}_E, |a_n\varpi_E^n| \leqslant 1, |a_n\varpi_E^n| \to 0 \text{ as } n \to -\infty\right\}.$$

X is a topologically nilpotent unit in R and so equipping R with the norm $|r| = \inf\{|\varpi_E|^n \mid r \in X^n R^\circ, n \in \mathbb{Z}\}$ makes R into a Banach–Tate $\mathbb{Z}_p$-algebra. Note that the restriction of this norm to $\mathcal{O}_E \subseteq R$ coincides with (the restriction of) the absolute value on E, so the fact that we are using the notation $|\cdot|$ should hopefully not cause any confusion. The norm $|\cdot|$ on R has the explicit description:

$$\left|\sum_{n\in\mathbb{Z}} a_n X^n\right| = \sup\{|a_n\varpi_E^n|\}. \tag{2.6.1}$$

Note that X is a multiplicative pseudouniformizer and there is no $x \in R$ with $1 < |x| < |X|^{-1}$.

LEMMA 2.6.2. *The norm we have defined on R is adapted to $\kappa_{(\eta,\omega)}$. Moreover, for $t \in T_1 = (1 + \varpi_v\mathcal{O}_{F,v})^2$ we have $|\kappa_{(\eta,\omega)}(t) - 1| \leqslant |\varpi_E|$.*

Proof. If $t \in T_1$ we have $\kappa_{(\eta,\omega)}(t) - 1 = u\zeta(1 + X)^\alpha - 1$ for some $\alpha \in \mathbb{Z}_p$, $u \in 1 + p\mathcal{O}_E$ and $\zeta \in \mathcal{O}_E$ a p-power root of unity. So $u\zeta \in 1 + \varpi_E\mathcal{O}_E$ and $|\kappa_{(\eta,\omega)}(t) - 1| \leqslant |\varpi_E|$. $\square$

We can now apply Lemma 2.6.1.

COROLLARY 2.6.3. *Consider the Fredholm series*

$$\det(1 - TU_v | H^0(K^v I_{v,1}, \mathcal{L}^v(k^v, w) \otimes_{\mathbb{Z}_p} \mathcal{D}_\kappa^{|\varpi_E|})) = \sum_{n \geqslant 0} c_n T^n \in R\{\{T\}\}.$$

Let

$$t = |D^\times \backslash (D \otimes_F \mathbb{A}_{F,f})^\times / K^v I_{v,1}| \prod_{v' | p, v' \neq v} (k_{v'} - 1).$$

(1) *We have* $c_n = \sum_{m \geqslant 0} b_{n,m} X^m \in \Lambda$ *for all* n.

(2) *Moreover,* $|b_{n,m} \varpi_E^m| \leqslant |\varpi_E^{\lambda(n)}|$ *for all* $m, n \geqslant 0$, *where* $\lambda(0) = 0, \lambda(1), \ldots$ *is a sequence of integers determined by*

$$\lambda(0) = 0, \lambda(i+1) = \lambda(i) + \lfloor i/t \rfloor - \lfloor i/pt \rfloor.$$

(3) *For* $z \in \mathbb{C}_p$ *with* $0 < v_p(z) < v_p(\varpi_E)$, *we have* $v_p(c_n(z)) \geqslant \lambda(n) v_p(z)$ *for every* $n \geqslant 0$, *with equality holding if and only if* $b_{n,\lambda(n)} \in \mathcal{O}_E^\times$. *If* $b_{n,\lambda(n)} \notin \mathcal{O}_E^\times$, *then*

$$v_p(c_n(z)) \geqslant \lambda(n) v_p(z) + \min\{v_p(z), v_p(\varpi_E) - v_p(z)\}.$$

Proof. The first two parts follow from Lemma 2.6.1, exactly as in [**JN19b**, Theorem 6.3.2]. Note that $n(|\varpi_E|, X, \lfloor i/t \rfloor) = \lfloor i/t \rfloor$ and $n(|\varpi_E|^{1/p}, X, \lfloor i/t \rfloor) = \lfloor p^{-1} \lfloor i/t \rfloor \rfloor = \lfloor i/pt \rfloor$.

Let $z \in \mathbb{C}_p$ with $0 < v_p(z) < v_p(\varpi_E)$. It follows immediately from the second part that $v_p(c_n(z)) \geqslant \lambda(n) v_p(z)$ for every $n \geqslant 0$, with equality holding if and only if $b_{n,\lambda(n)} \in \mathcal{O}_E^\times$. Finally, if $b_{n,\lambda(n)} \notin \mathcal{O}_E^\times$ then $v_p(b_{n,\lambda(n)}) \geqslant v_p(\varpi_E)$ and the rest of the third part is easy to check. □

2.7. The spectral curve and partial eigenvariety. We can now construct the spectral curve $\mathcal{Z}^{U_v}(k^v, w) \to \mathcal{W}$ for the compact operator U_v acting on the spaces $H^0(K^v I_{v,1}, \mathcal{L}^v(k^v, w) \otimes_{\mathbb{Z}_p} \mathcal{D}_\kappa^r)$, for $\kappa = \kappa_U$ ranging over the weights induced from affinoid open $U \subseteq \mathcal{W}$, as well as the pullback $\mathcal{Z}^{U_v}(k^v, w)_{\eta,\omega}^{>r} \to \mathcal{W}_{\eta,\omega}^{>r}$ for $r \in (0,1) \cap \mathbb{Q}$ and $\mathcal{W}_{\eta,\omega}$ a component of weight space.

PROPOSITION 2.7.1. *Fix a component* $\mathcal{W}_{\eta,\omega}$ *of weight space, and let* $E = E_\eta = \mathbb{Q}_p(\zeta_{p^c}) \subset \mathbb{C}_p$ *be the subfield generated by the image of* η *as before. Set* $s_0 = 1$ *if* w *is even and* $s_0 = 2$ *if* w *is odd. Then*

$$\mathcal{Z}^{U_v}(k^v, w)_{\eta,\omega}^{>|\varpi_E|} = \mathcal{Z}_0 \sqcup \mathcal{Z}_{(0,s_0)} \sqcup \coprod_{\substack{k \geqslant 2 \\ k \equiv w \bmod 2}} \mathcal{Z}_{k-1} \sqcup \mathcal{Z}_{(k-1,k+1)}$$

 https://doi.org/10.1017/fms.2019.23

is a disjoint union of rigid analytic spaces $\mathcal{Z}_I$ (with I denoting an interval as in the displayed formula) which are finite and flat over $\mathcal{W}_{\eta,\omega}^{>|\varpi_E|}$. For each point $x \in \mathcal{Z}_I$, with corresponding U_v-eigenvalue λ_x, we have $v_p(\lambda_x) \in (p-1)p^c v_p(z_{\kappa(x)}) \cdot I$.

Note that s_0 is the highest slope appearing in the spaces of classical automorphic forms with minimal weight, under the restriction that the weight has the same parity as w.

Proof. This follows from Corollary 2.6.3 and Lemma 2.4.1, as in [**LWX17**, Proof of Theorem 1.3]. We sketch the argument. Recall the finite order character $\tilde{\eta}$ associated with η from Remark 2.5.2, and note that $\tilde{\eta}$ factors through $(\mathcal{O}_{F,v}/\varpi^{c+1})^\times$ (if $c \geqslant 1$, the conductor of $\tilde{\eta}$ is (ϖ^{c+1}); if $c = 0$, $\tilde{\eta}$ is either tame or trivial). By passing to a normal compact open subgroup of K^v we may assume that

$$K_{v'} \subset \begin{pmatrix} 1+\varpi_{v'}^{c+2}\mathcal{O}_{F,v'} & \mathcal{O}_{F,v'} \\ \varpi_{v'}^{c+2}\mathcal{O}_{F,v'} & 1+\varpi_{v'}^{c+2}\mathcal{O}_{F,v'} \end{pmatrix}$$

for each $v'|p$, $v' \neq v$. Now we consider points $\kappa \in \mathcal{W}(\mathbb{C}_p)$ which are locally algebraic of weight (k, w) and character ϵ, with the ϵ_i factoring through $(\mathcal{O}_{F,v}/\varpi^{c+2})^\times$ and ϵ_2/ϵ_1 of conductor (ϖ^{c+2}). We furthermore insist that κ is in the component $\mathcal{W}_{\eta,\omega}$. This amounts to requiring that $\kappa|_{\Delta\times\Delta} = \omega$ and $\tilde{\eta} = \epsilon_1\epsilon_2$. We have

$$v_p(z_\kappa) = v_p(\exp(p)^{k-2}(\epsilon_1/\epsilon_2)(\exp(p)) - 1) = 1/\phi(p^{c+1}) < v_p(\varpi_E) = 1/\phi(p^c)$$

and hence $\kappa \in \mathcal{W}_{\eta,\omega}^{>|\varpi_E|}(\mathbb{C}_p)$.

For $k \in \mathbb{Z}_{\geqslant 2}$ with $k \equiv w \pmod 2$ we set $n_k = (k-1)p^{c+1}t = \dim S(k, w, \epsilon)$, where

$$t = |D^\times \backslash (D \otimes_F \mathbb{A}_{F,f})^\times / K^v I_{v,1}| \prod_{v'|p, v' \neq v} (k_{v'} - 1)$$

as in the beginning of Section 2.4. We now carry out 'Step I' of [**LWX17**, Proof of Theorem 1.3]: this identifies certain points on the Newton polygon of the characteristic power series $\sum_{n\geqslant 0} c_n(z_\kappa)T^n$ of U_v at the weight κ. We have

$$\begin{aligned} v_p(z_\kappa)\lambda(n_k) &= \frac{1}{\phi(p^{c+1})} \sum_{i=0}^{(k-1)p^{c+1}t-1} \left(\left\lfloor \frac{i}{t} \right\rfloor - \left\lfloor \frac{i}{pt} \right\rfloor \right) \\ &= \frac{1}{\phi(p^{c+1})} \left(t \sum_{i=0}^{(k-1)p^{c+1}-1} i - pt \sum_{i=0}^{(k-1)p^c-1} i \right) \end{aligned}$$

$$= \frac{(k-1)^2 p^{c+1} t}{2}.$$

So, by Lemma 2.4.1, $v_p(z_\kappa)\lambda(n_k)$ is equal to half the sum of the U_v-slopes on $S(k, w, \epsilon)$ and $S(k, w, \epsilon^{-1})$. Combining this with Corollary 2.6.3(3) (at the weights corresponding to both (k, w, ϵ) and (k, w, ϵ^{-1})), we deduce that the sum of the first n_k U_v-slopes on $H^0(K^v I_{v,1}, \mathcal{L}^v(k^v, w) \otimes_{\mathbb{Z}_p} \mathcal{D}_\kappa^{|\varpi_E|})$ is $((k-1)^2 p^{c+1} t)/2$ and that the Newton polygon of $\sum_{n \geqslant 0} c_n(z_\kappa) T^n$ passes through the point $\mathscr{P}_k = (n_k, \lambda(n_k) v_p(z_\kappa))$.

'Step II' of [LWX17, Proof of Theorem 1.3] can now be carried over (replacing q with p^{c+1}), and this establishes the rest of the proposition. We denote the first coordinate of the vertices of the Newton polygon either side of $\mathscr{P}_k$ by n_k^- and n_k^+ (if $\mathscr{P}_k$ is itself a vertex we have $n_k^- = n_k^+ = n_k$). If $n_k^- \neq n_k^+$ then the slope of the segment containing $\mathscr{P}_k$ is $(k-1)$ (since this is the slope of the lower bound Newton polygon) and $n_k^- \in [n_k - t, n_k]$, $n_k^+ \in [n_k, n_k + t]$. Whether or not n_k^- equals n_k^+, it now follows from Corollary 2.6.3(3) that n_k^- is the minimal index i in $[n_k - t, n_k]$ with $b_{i,\lambda(i)} \in \mathcal{O}_E^\times$ and n_k^+ is the maximal index i in $[n_k, n_k + t]$ with $b_{i,\lambda(i)} \in \mathcal{O}_E^\times$. If we specialize to $z \in \mathcal{W}_{\eta,\omega}^{>|\varpi_E|}$ we have, for all $i \geqslant 0$:

$$v_p(c_{n_k - i}(z)) \geqslant v_p(z)\lambda(n_k - i) \geqslant v_p(z)(\lambda(n_k) - (k-1)\phi(p^{c+1})i)$$

where the first equality is strict if $n_k - t \leqslant n_k - i < n_k^-$ (minimality of n_k^-) and the second inequality is strict if $n_k - i < n_k - t$. These strict inequalities have difference between the two sides at least $\min\{v_p(z), v_p(\varpi_E) - v_p(z)\}$. A similar inequality holds for $n_k + i > n_k^+$. We get equalities when $n_k - i = n_k^-$ and $n_k + i = n_k^+$. We deduce that, if $n_k^- \neq n_k^+$ then $(n_k^-, \lambda(n_k^-) v_p(z))$ and $(n_k^+, \lambda(n_k^+) v_p(z))$ are consecutive vertices of the Newton polygon of $\sum_{n \geqslant 0} c_n(z) T^n$ for all $z \in \mathcal{W}_{\eta,\omega}^{>|\varpi_E|}$. The slope of the segment connecting these vertices is $(k-1)\phi(p^{c+1}) v_p(z)$ and it therefore passes through $(n_k, \lambda(n_k) v_p(z))$. If $n_k^+ = n_k^- = n_k$ then $(n_k, \lambda(n_k) v_p(z))$ is a vertex of the Newton polygon for all $z \in \mathcal{W}_{\eta,\omega}^{>|\varpi_E|}$. Now for an interval I as in the statement of the proposition we define $\mathcal{Z}_I$ to be the (open) subspace of $\mathcal{Z}^{U_v}(k^v, w)_{\eta,\omega}^{>|\varpi_E|}$ given by demanding that the slope lies in $(p-1)p^c v_p(z_{\kappa(x)}) \cdot I$. Each $\mathcal{Z}_I$ pulls back to an affinoid open of (the pullback of) $\mathcal{Z}^{U_v}(k^v, w)_{\eta,\omega}^{>|\varpi_E|}$ over every affinoid open of $\mathcal{W}_{\eta,\omega}^{>|\varpi_E|}$ (when I is an open interval we use the bound in terms of $\min\{v_p(z), v_p(\varpi_E) - v_p(z)\}$ to see that we still get an affinoid open). By our control over the Newton polygons (or Hida theory for $\mathcal{Z}_0$), it follows from [Buz07, Corollary 4.3] that the $\mathcal{Z}_I$ are finite flat over $\mathcal{W}_{\eta,\omega}^{>|\varpi_E|}$. □

REMARK 2.7.2. (1) In the above Proposition, we are proving that the partial eigenvariety exhibits 'halo' behaviour over a boundary annulus whose radius depends on the conductor of the character $\tilde{\eta}$. This dependence may well be an artefact of the method of proof.

(2) The rest of the arguments in [LWX17] also generalize to this situation, showing that the slopes over a sufficiently small boundary annulus in weight space are a union of finitely many arithmetic progressions with the same common difference.

(3) Another way to obtain one-dimensional families of p-adic automorphic forms for the quaternion algebra D is to allow the weight to vary in the parallel direction. The methods described here do not seem sufficient to establish 'halo' behaviour for these one-dimensional families—one reason is that there is no longer a sharp numerical classicality theorem in terms of a single compact operator in this case.

Now we fix an integer $n \geqslant 1$ and assume that $K_{v'} = I'_{v',1,n-1}$ for each place $v'|p$ with $v' \neq v$. We set $K = K^v I_{v,1}$. Let S be the set of finite places w of F where either $w|p$, D_w is nonsplit or D_w is split but $K_w \neq \mathcal{O}_{D,w}^\times$. For $w \notin S$ we have Hecke operators

$$S_w = \left[K \begin{pmatrix} \varpi_w & 0 \\ 0 & \varpi_w \end{pmatrix} K \right], \quad T_w = \left[K \begin{pmatrix} 1 & 0 \\ 0 & \varpi_w \end{pmatrix} K \right],$$

which are independent of the choice of uniformizer $\varpi_w \in F_w$.

The spaces $H^0(K, \mathcal{L}^v(k^v, w) \otimes_{\mathbb{Z}_p} \mathcal{D}_\kappa^r)$ give rise to a coherent sheaf $\mathcal{H}$ over $\mathcal{Z}^{U_v}(k^v, w)$, equipped with an action of the Hecke operators $\{U_{v'}, v'|p\}$ and $\{S_w, T_w : w \notin S\}$. If we let $\mathbb{T}$ denote the free commutative $\mathbb{Z}_p$-algebra generated by these Hecke operators, and let $\psi : \mathbb{T} \to \mathrm{End}(\mathcal{H})$ be the map induced by the Hecke action, then we have an eigenvariety datum $(\mathcal{W} \times \mathbb{A}^1, \mathcal{H}, \mathbb{T}, \psi)$. Here we use the notion of eigenvariety datum as defined in [JN19a, Section 3.1], and we refer to [JN19a, Section 3.1] for the construction of the eigenvariety associated with an eigenvariety datum (we remark that we could also have used the eigenvariety $(\mathcal{Z}^{U_v}(k^v, w), \mathcal{H}, \mathbb{T}, \psi)$, since $\mathcal{H}$ is supported on $\mathcal{Z}^{U_v}(k^v, w)$). We denote the associated eigenvariety by $\mathcal{E}(k^v, w)$ and, if $r \in (0, 1)$, denote its pullback to $\mathcal{W}^{>r}$ by $\mathcal{E}(k^v, w)^{>r}$.

DEFINITION 2.7.3. A *classical point* of $\mathcal{E}(k^v, w)$ is a point with locally algebraic weight corresponding to a Hecke eigenvector with nonzero image under the map π of (2.3.1), whose systems of Hecke eigenvalue do not arise from one-dimensional automorphic representations of $(D \otimes_F \mathbb{A}_F)^\times$.

The points excluded in the above definition do not correspond to classical Hilbert modular forms under the Jacquet–Langlands correspondence. They all have parallel weight 2, and their $U_{v'}$ eigenvalues $\alpha_{v'}$ satisfy $v_p(\alpha_{v'}) = 1$ for all $v'|p$.

PROPOSITION 2.7.4.

(1) *$\mathcal{E}(k^v, w)$ is equidimensional of dimension 1 and flat over $\mathcal{W}$.*

(2) *Let C be an irreducible component of $\mathcal{E}(k^v, w)$ and let $k \geqslant 2$ be an integer with the same parity as w. Then C contains a point which is locally algebraic of weight (k, w) and U_v-slope $< k-1$ (in particular, this point is classical).*

Proof. (1) This part is proved in the same way as the well-known analogous statement for the Coleman–Mazur eigencurve (for example, see [JN19b, Section 6.1]).

(2) This is a consequence of Proposition 2.7.1. Indeed, C maps to a single component $\mathcal{W}_{\eta,\omega}$ of weight space, and we let $E = \mathbb{Q}_p(\zeta^c)$ be as in that proposition. Then any irreducible component C' of $C^{>|\varpi_E|}$ is finite flat over an irreducible component of $\mathcal{Z}^{U_v}(k^v, w)^{>|\varpi_E|}$. It follows from Proposition 2.7.1 that C' is finite flat over $\mathcal{W}_{\eta,\omega}^{>|\varpi_E|}$ and there is an interval I such that for each point $x \in C'$, with corresponding U_v-eigenvalue λ_x, we have $v_p(\lambda_x) \in (p-1)p^c v_p(z_{\kappa(x)}) \cdot I$. If $v_p(z_{\kappa(x)})$ is sufficiently small, then $v_p(\lambda_x) < k-1$. To finish the proof, we note that we can find a locally algebraic point of weight (k, w) in $\mathcal{W}_{\eta,\omega}^{>|\varpi_E|}$ with $v_p(z_{\kappa(x)})$ as small as we like, by choosing a character $\epsilon = (\epsilon_1, \epsilon_2)$ (compatible with η and ω) such that $\epsilon_1(\exp(p))/\epsilon_2(\exp(p))$ is a p-power root of unity of sufficiently large order. □

REMARK 2.7.5. The existence of partial eigenvarieties (where weights vary above a proper subset of the places dividing p) is probably well known to experts, but they have not been utilized so much in the literature. They were discussed (for definite unitary groups) and used in works of Chenevier [Che] and Chenevier–Harris [CH13] to remove regularity hypotheses from theorems concerning the existence and local–global compatibility of Galois representations associated to automorphic representations. They can be viewed as eigenvarieties defined with respect to certain nonminimal parabolic subgroups of the relevant group (see [Loe11])—to recover the setting of this article we let $P \subset (\mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{GL}_2)_{\mathbb{Q}_p} = \prod_{v'|p} \mathrm{Res}_{F_{v'}/\mathbb{Q}_p}\,\mathrm{GL}_2$ be the parabolic with component labelled by our fixed place v equal to the upper triangular Borel and other components equal to the whole group GL_2. See, for example, [Din18] for another application of eigenvarieties defined with respect to a nonminimal parabolic. We also note that partial Hilbert modular eigenvarieties have been used recently by Barrera, Dimitrov and Jorza to study the exceptional zero conjecture for Hilbert modular forms [BDJ17].

3. The parity conjecture

We can now apply the preceding results to establish some new cases of the parity conjecture for Hilbert modular forms, using [PX, Theorems A and B]. First we need to discuss the family of Galois representations carried by the partial eigenvarieties and their p-adic Hodge theoretic properties.

3.1. Galois representations.

We begin by noting that there is a continuous 2-dimensional pseudocharacter $T : G_F \to \mathcal{O}_{\mathcal{E}(k^v,w)}$ with $T(\mathrm{Frob}_w) = T_w$ for all $w \notin S$. This follows from [Che04, Proposition 7.1.1], the Zariski density of classical points in $\mathcal{E}(k^v, w)$ and the existence of Galois representations associated to Hilbert modular forms.

We denote the normalization of $\mathcal{E}(k^v, w)$ by $\tilde{\mathcal{E}}$, and we also denote by T the pseudocharacter on $\tilde{\mathcal{E}}$ obtained by pullback from $\mathcal{E}(k^v, w)$. We denote by $\tilde{\mathcal{E}}^{\mathrm{irr}} \subset \tilde{\mathcal{E}}$ the (Zariski open, see [Che14, Example 2.20]) locus where T is absolutely irreducible. Note also that $\tilde{\mathcal{E}}^{\mathrm{irr}}$ is Zariski-dense in $\tilde{\mathcal{E}}$, since classical points have irreducible Galois representations.

PROPOSITION 3.1.1. *There is a locally free rank 2 $\mathcal{O}_{\tilde{\mathcal{E}}^{\mathrm{irr}}}$-module V equipped with a continuous representation*

$$\rho : G_{F,S} \to \mathrm{GL}_{\mathcal{O}_{\tilde{\mathcal{E}}^{\mathrm{irr}}}}(V)$$

satisfying

$$\det(X - \rho(\mathrm{Frob}_w)) = X^2 - T_w X + q_w S_w$$

for all $w \notin S$ (where Frob_w *denotes an arithmetic Frobenius).*

Proof. First we recall that T canonically lifts to a continuous representation $\rho : G_{F,S} \to \mathcal{A}^\times$ where $\mathcal{A}$ is an Azumaya algebra of rank 4 over $\tilde{\mathcal{E}}^{\mathrm{irr}}$, by [Che04, Corollary 7.2.6] (see also [Che14, Proposition G]). It remains to show that $\mathcal{A}$ is isomorphic to the endomorphism algebra of a vector bundle.

On the other hand, [CM98, Theorem 5.1.2] and the remark appearing after this theorem show that there is an admissible cover of $\tilde{\mathcal{E}}^{\mathrm{irr}}$ by affinoid opens $\{U_i\}_{i\in I}$, together with continuous representations

$$\rho_i : G_{F,S} \to \mathrm{GL}_2(\mathcal{O}(U_i))$$

lifting the pseudocharacters $T|_{U_i}$.

By the uniqueness part of [Che04, Lemma 7.2.4], we deduce that for each affinoid open U_i, we have an isomorphism of $\mathcal{O}(U_i)$-algebras

$\mathcal{A}(U_i) \cong M_2(\mathcal{O}(U_i))$. Now, by the standard argument relating the Brauer group to cohomology (see [Mil80, Theorem IV.2.5]; the constructions using either Čech cohomology or gerbes can be applied, since Čech cohomology coincides with usual cohomology on $\tilde{\mathcal{E}}^{\text{irr}}$ by [vdP82, Proposition 1.4.4]) we can associate to $\mathcal{A}$ an element $F_{\mathcal{A}}$ of $H^2(\tilde{\mathcal{E}}^{\text{irr}}, \mathcal{O}^{\times}_{\tilde{\mathcal{E}}^{\text{irr}}})$ (crucially, this is the cohomology on the rigid analytic site, not étale cohomology). This element is trivial if and only if $\mathcal{A}$ is isomorphic to the endomorphism algebra of a vector bundle.

Since $\tilde{\mathcal{E}}^{\text{irr}}$ is separated and one-dimensional, $H^2(\tilde{\mathcal{E}}^{\text{irr}}, \mathcal{F})$ vanishes for any Abelian sheaf $\mathcal{F}$ by [dJvdP96, Corollary 2.5.10, Remark 2.5.11]. In particular, $F_{\mathcal{A}}$ is trivial and we are done. □

REMARK 3.1.2. In fact, [CM98, Theorem 5.1.2] applies over the whole of $\tilde{\mathcal{E}}$, not just the irreducible locus, so there are representations $\rho_i : \mathcal{O}(U_i)[G_{F,S}] \to M_2(\mathcal{O}(U_i))$ lifting T over each member of an admissible affinoid cover $\{U_i\}_{i\in I}$ of $\tilde{\mathcal{E}}$. Shrinking the cover if necessary, we may assume that the intersection of any two distinct covering affinoids $U_i \cap U_j$ is contained in $\tilde{\mathcal{E}}^{\text{irr}}$, so we obtain a canonical isomorphism of $\mathcal{O}_{U_i\cap U_j}$-algebras $M_2(\mathcal{O}_{U_i\cap U_j}) \cong M_2(\mathcal{O}_{U_i\cap U_j})$ intertwining the representations ρ_i and ρ_j.

This gives the gluing data necessary to define a Galois representation to an Azumaya algebra, which as above is isomorphic to the endomorphism algebra of a vector bundle (by the vanishing of $H^2(\tilde{\mathcal{E}}, \mathcal{O}^{\times}_{\tilde{\mathcal{E}}})$). So we finally obtain a (possibly noncanonical) Galois representation on a vector bundle over $\tilde{\mathcal{E}}$, although in what follows we will just work over the irreducible locus as this suffices for our purposes.

If z is a point of $\tilde{\mathcal{E}}^{\text{irr}}$ we denote by V_z the specialization of V at z, which we regard as a 2-dimensional representation of G_F over the residue field $k(z)$. Note that we obtain a continuous character $\det\rho : G_F \to \Gamma(\tilde{\mathcal{E}}^{\text{irr}}, \mathcal{O}_{\tilde{\mathcal{E}}^{\text{irr}}})^{\times}$ with $\det\rho(z) = \det V_z$. In fact $\det\rho$ is constant on connected components of $\tilde{\mathcal{E}}^{\text{irr}}$. Indeed, for every classical point z, $\det\rho(z)$ is Hodge–Tate of weight $-1-w$ at every place dividing p, and the results cited in the proof of Proposition 3.1.8 below show that this property extends to all points z. Class field theory and the fact that Hodge–Tate characters are locally algebraic [Ser98, Theorem 3, Appendix to Ch. III] shows that each $\det\rho(z)$ is the product of a finite order character and the $(1+w)$-power of the cyclotomic character, so $\det\rho$ is constant on connected components. Before proceeding we fix some conventions on local class field theory. If γ an element of some field, we recall that we have denoted by $ur(\gamma)$ the character on $F_v^{\times}$ which is trivial on $\mathcal{O}_{F_v}^{\times}$ and takes uniformizers to γ. We will freely interpret characters of $F_v^\times$ as characters of W_{F_v} via the Artin reciprocity map, normalized to take a uniformizer to a geometric Frobenius.

We use covariant Dieudonné modules so the cyclotomic character has Hodge–Tate weight -1.

PROPOSITION 3.1.3. *Let z be a classical point of $\tilde{\mathcal{E}}^{\mathrm{irr}}$, locally algebraic of weight (k, w) and character ϵ, and with U_v-eigenvalue α. Set $\widetilde{\alpha} = \alpha p^{1+(w-k)/2}$. Then $V_z|_{G_{F_v}}$ is potentially semistable with Hodge–Tate weights $(-(w+k)/2, -(1+(w-k)/2))$, and the associated (Frobenius-semisimplified) Weil–Deligne representation is of the form*

$$WD(V_z|_{G_{F_v}})^{F-ss} = \big(ur(\widetilde{\beta})^{-1}\epsilon_1 \oplus ur(\widetilde{\alpha})^{-1}\epsilon_2, N\big)$$

for some $\widetilde{\beta} \in \mathbb{C}_p$ (determined by the determinant of V_z) which satisfies $v_p(\widetilde{\alpha}) + v_p(\widetilde{\beta}) = 1 + w$. N is nonzero if and only if $\epsilon_1 = \epsilon_2$ and $\widetilde{\alpha}/\widetilde{\beta} = p^{-1}$.

Proof. This follows from the local–global compatibility theorem of Saito [Sai09], as completed by Skinner [Ski09] and Liu [Liu12]. Note that α is an eigenvalue for the $\star$-action of U_v on $S(k, w, \epsilon)$ so $\widetilde{\alpha}$ is the corresponding eigenvalue for the standard action of U_v. □

DEFINITION 3.1.4. We say that a classical point z of weight (k, w) is *critical* if $V_z|_{G_{F_v}}$ is decomposable and $v_p(U_v(z)) = k - 1$.

REMARK 3.1.5. It follows from weak admissibility that $WD(V_z|_{G_{F_v}})$ fails to be Frobenius-semisimple if and only if the two characters $ur(\widetilde{\beta})^{-1}\epsilon_1, ur(\widetilde{\alpha})^{-1}\epsilon_2$ are equal (in which case we also have $N = 0$).

In what remains of this section we give details about the triangulations of the p-adic Galois representations for noncritical classical points of $\tilde{\mathcal{E}}^{\mathrm{irr}}$ and apply the results of [KPX14] to establish the existence of global triangulations for the family of Galois representations over this eigenvariety. Everything here should be unsurprising to experts, but we have written out some details because in the literature it is common to restrict to the semistable case, whilst it is crucial for us to consider classical points which are only potentially semistable.

3.1.1. Triangulation at v for classical points. Suppose $z \in \tilde{\mathcal{E}}^{\mathrm{irr}}(L)$ is a noncritical classical point (with $L/\mathbb{Q}_p$ finite), locally algebraic of weight (k, w) and character ϵ, and with U_v-eigenvalue α. Suppose moreover that $V_z|_{G_{F_v}}$ is indecomposable and potentially crystalline. It follows from Proposition 3.1.3 that this representation becomes crystalline over an Abelian extension of F_v.

So the L-Weil–Deligne representation $WD(V_z|_{G_{F_v}})$ has an admissible filtration after extending scalars to the $\mathbb{Q}_p$-algebra $L_\infty = \bigcup_N L \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\zeta_{p^N})$ (see [Col08, Section 4.4]). Moreover, we can explicitly describe this admissible filtered Weil–Deligne module D_z as in [Col08, Section 4.5]. Indeed, when the action of W_{F_v} on D_z is semisimple, we let e_1 and e_2 be basis vectors with W_{F_v} action given by $ur(\widetilde{\beta})^{-1}\epsilon_1$ and $ur(\widetilde{\alpha})^{-1}\epsilon_2$, respectively. When the action of W_{F_v} is not semisimple, we have $ur(\widetilde{\beta})^{-1}\epsilon_1 = ur(\widetilde{\alpha})^{-1}\epsilon_2$ and there are basis vectors e_1, e_2 with W_{F_v} action given by $\sigma e_1 = ur(\widetilde{\beta})^{-1}\epsilon_1(\sigma)e_1$ and $\sigma e_2 = ur(\widetilde{\beta})^{-1}\epsilon_1(\sigma)(e_2 - (\deg(\sigma))e_1)$ (σ maps to the $\deg(\sigma)$ power of arithmetic Frobenius in the absolute Galois group of the residue field). The filtration is given by

$$\mathrm{Fil}^i(L_\infty \otimes_L D_z) = \begin{cases} 0 & \text{if } i > -\left(1 + \dfrac{w-k}{2}\right) \\ L_\infty \cdot (\gamma e_1 + e_2) & \text{if } -\dfrac{w+k}{2} < i \leqslant -\left(1 + \dfrac{w-k}{2}\right) \\ L_\infty \otimes_L D_z & \text{if } i \leqslant -\dfrac{w+k}{2} \end{cases}$$

where $\gamma \in L_\infty$ is the (invertible) value of a certain Gauss sum in the semisimple case and $\gamma = 0$ in the nonsemisimple case.

It follows from [Col08, Proposition 4.13] that $V_z|_{G_{F_v}}$ is trianguline. To describe the triangulations we adopt the notation of [Col08]. So $\mathscr{R}$ denotes the Robba ring over L and if $\delta : \mathbb{Q}_p^\times \to L^\times$ is a continuous character $\mathscr{R}(\delta)$ denotes that (ϕ, Γ)-module obtained by multiplying the action of ϕ on $\mathscr{R}$ by $\delta(p)$ and the action of $\gamma \in \Gamma = \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$ by $\delta(\chi_{\mathrm{cyc}}(\gamma))$. If V is an L-representation of $G_{\mathbb{Q}_p}$ we denote by $\mathbf{D}_{\mathrm{rig}}(V)$ the slope zero (ϕ, Γ)-module over $\mathscr{R}$ associated to V by [Col08, Proposition 1.7].

Now we have recalled the necessary notation we can recall the precise statement of [Col08, Proposition 4.13]: we have two triangulations (they coincide in the nonsemisimple case)

$$0 \to \mathscr{R}(x^{(w+k)/2} ur(\widetilde{\beta})^{-1}\epsilon_1) \to \mathbf{D}_{\mathrm{rig}}(V_z) \to \mathscr{R}(x^{(1+(w-k)/2)} ur(\widetilde{\alpha})^{-1}\epsilon_2) \to 0$$
$$0 \to \mathscr{R}(x^{(w+k)/2} ur(\widetilde{\alpha})^{-1}\epsilon_2) \to \mathbf{D}_{\mathrm{rig}}(V_z) \to \mathscr{R}(x^{(1+(w-k)/2)} ur(\widetilde{\beta})^{-1}\epsilon_1) \to 0.$$

We can rewrite the first of these triangulations as

$$0 \to \mathscr{R}(\delta^{-1} \det \rho(z)) \to \mathbf{D}_{\mathrm{rig}}(V_z) \to \mathscr{R}(\delta(z)) \to 0$$

where we define the continuous character $\delta : F_v^\times \to \Gamma(\tilde{\mathscr{E}}^{\mathrm{irr}}, \mathcal{O}_{\tilde{\mathscr{E}}^{\mathrm{irr}}})^\times$ by $\delta(p) = U_v^{-1}$ and $\delta|_{\mathcal{O}_{F_v}^\times} = \kappa|_{\{1\}\times\mathcal{O}_{F_v}^\times}$.

Next, we consider the case where $V_z|_{G_{F_v}}$ is not potentially crystalline. Following [Col08, Section 4.6] we can describe the associated admissible filtered Weil–Deligne module D_z: letting e_2 and e_1 be basis vectors with W_{F_v} action given by

$ur(p\widetilde{\alpha})^{-1}\epsilon_2$ and $ur(\widetilde{\alpha})^{-1}\epsilon_2$, respectively, we have $Ne_1 = e_2$, $Ne_2 = 0$, and the filtration is given by

$$\mathrm{Fil}^i(L_\infty \otimes_L D_z) = \begin{cases} 0 & \text{if } i > -\left(1 + \dfrac{w-k}{2}\right) \\ L_\infty \cdot (e_1 - \mathscr{L} e_2) & \text{if } -\dfrac{w+k}{2} < i \leqslant -\left(1+\dfrac{w-k}{2}\right) \\ L_\infty \otimes_L D_z & \text{if } i \leqslant -\dfrac{w+k}{2} \end{cases}$$

for some $\mathscr{L} \in L$.

It follows from [**Col08**, Proposition 4.18] that $V_z|_{G_{F_v}}$ is trianguline. Indeed, we have a triangulation

$$0 \to \mathscr{R}(x^{(w+k)/2}|x|ur(\widetilde{\alpha})^{-1}\epsilon_2) \to \mathbf{D}_{\mathrm{rig}}(V_z) \to \mathscr{R}(x^{(1+(w-k)/2)}ur(\widetilde{\alpha})^{-1}\epsilon_2) \to 0$$

which again can be rewritten as

$$0 \to \mathscr{R}(\delta^{-1}\det\rho(z)) \to \mathbf{D}_{\mathrm{rig}}(V_z) \to \mathscr{R}(\delta(z)) \to 0.$$

The final case we have to consider is when $V_z|_{G_{F_v}}$ is decomposable. Then we have $v_p(\alpha) = 0$ (by the noncritical assumption on z) and

$$\mathbf{D}_{\mathrm{rig}}(V_z) = \mathscr{R}(x^{(w+k)/2}ur(\widetilde{\beta})^{-1}\epsilon_1) \oplus \mathscr{R}(x^{(1+(w-k)/2)}ur(\widetilde{\alpha})^{-1}\epsilon_2).$$

We can summarize our discussion in the following:

PROPOSITION 3.1.6. *Suppose $z \in \tilde{\mathcal{E}}^{\mathrm{irr}}$ is a noncritical classical point. Then there is a triangulation*

$$0 \to \mathscr{R}(\delta^{-1}\det\rho(z)) \to \mathbf{D}_{\mathrm{rig}}(V_z) \to \mathscr{R}(\delta(z)) \to 0$$

where we define the continuous character $\delta : F_v^\times \to \Gamma(\tilde{\mathcal{E}}^{\mathrm{irr}}, \mathcal{O}_{\tilde{\mathcal{E}}^{\mathrm{irr}}})^\times$ by $\delta(p) = U_v^{-1}$ and $\delta|_{\mathcal{O}_{F_v}^\times} = \kappa|_{\{1\}\times\mathcal{O}_{F_v}^\times}$. Moreover, this triangulation is strict *in the sense of* [**KPX14**, *Definition* 6.3.1].

Proof. Let $D = \mathbf{D}_{\mathrm{rig}}(V_z) \otimes_{\mathscr{R}} \mathscr{R}(\delta^{-1}\det\rho)^{-1}(z)$. The only thing remaining to be verified is that the triangulation is strict, which comes down to checking that $D^{\phi=1,\Gamma=1}$ is one-dimensional over the coefficient field L. Suppose for a contradiction that this space has dimension 2. Then, by [**Col08**, Proposition 2.1] we have $(\delta^{-1}\det\rho)(z) = x^i\delta(z)$ for some $i \in \mathbb{Z}_{\geqslant 0}$. Comparing these characters on the intersection of the kernels of ϵ_1 and ϵ_2 we deduce that $i = k-1$. We have $(\delta^{-1}\det\rho)(z) = x^{k-1}\delta(z)$ if and only if the characters $ur(\widetilde{\beta})^{-1}\epsilon_1$, $ur(\widetilde{\alpha})^{-1}\epsilon_2$

appearing in D_z are equal. It follows from the correspondence between filtered Weil–Deligne modules and potentially semistable (ϕ, Γ)-modules established by [Ber08, Theorem A] that $D[1/t]^{\Gamma=1}$ is two-dimensional with nontrivial unipotent ϕ action, so $D[1/t]^{\phi=1,\Gamma=1}$ is one-dimensional which gives the desired statement. $\square$

We can now apply the results of [KPX14] to establish the existence of a global triangulation for the family of Galois representations V. In the below statement we adopt the notation of [KPX14, 6.2.1, 6.2.2] so for a rigid space $X/\mathbb{Q}_p$ and a character $\delta_X : \mathbb{Q}_p^\times \to \Gamma(X, \mathcal{O}_X)^\times$ we have a free rank one (ϕ, Γ)-module $\mathscr{R}_X(\delta_X)$ over the sheaf of Robba rings $\mathscr{R}_X$. We also have a rank two (ϕ, Γ)-module $\mathbf{D}_{\mathrm{rig}}(V|_{G_{F_v}})$ over $\mathscr{R}_{\tilde{\mathcal{E}}^{\mathrm{irr}}}$.

COROLLARY 3.1.7.

(1) $\mathbf{D}_{\mathrm{rig}}(V|_{G_{F_v}})$ *is densely pointwise strictly trianguline in the sense of* [KPX14, *Definition* 6.3.2], *with respect to the ordered parameters* $\delta^{-1} \det \rho, \delta$ *and the Zariski-dense subset of noncritical classical points.*

(2) *There are line bundles* $\mathscr{L}_1$ *and* $\mathscr{L}_2$ *over* $\tilde{\mathcal{E}}^{\mathrm{irr}}$ *and an exact sequence*

$$0 \to \mathscr{R}_{\tilde{\mathcal{E}}^{\mathrm{irr}}}(\delta^{-1} \det \rho) \otimes_{\mathcal{O}_{\tilde{\mathcal{E}}^{\mathrm{irr}}}} \mathscr{L}_1 \to \mathbf{D}_{\mathrm{rig}}(V|_{G_{F_v}}) \to \mathscr{R}_{\tilde{\mathcal{E}}^{\mathrm{irr}}}(\delta) \otimes_{\mathcal{O}_{\tilde{\mathcal{E}}^{\mathrm{irr}}}} \mathscr{L}_2$$

with the cokernel of the final map vanishing over a Zariski open subset which contains every noncritical classical point. In particular, this sequence induces a triangulation of $\mathbf{D}_{\mathrm{rig}}(V_z|_{G_{F_v}})$ *at every noncritical classical point.*

Proof. The first assertion follows immediately from Proposition 3.1.6. The existence of the global triangulation and the fact that the cokernel of the final map vanishes over a Zariski open subset containing every noncritical classical point follows from [KPX14, Corollary 6.3.10]. To apply this Corollary we have to check that there exists an admissible affinoid cover of $\tilde{\mathcal{E}}^{\mathrm{irr}}$ such that the noncritical classical points are Zariski-dense in each member of the cover—this property is the definition of 'Zariski-dense' in [KPX14, Definition 6.3.2]. We prefer to reserve the terminology Zariski-dense for the standard property that a set of points is not contained in a proper analytic subset. The (a priori) stronger density statement needed to apply [KPX14, Corollary 6.3.10] follows from [CHJ17, Lemma 5.9]. To apply this, we show that $\tilde{\mathcal{E}}^{\mathrm{irr}}$ has an increasing cover by affinoids as in [CHJ17, Remark 5.10]. Indeed, $\tilde{\mathcal{E}}$ has an increasing cover by affinoids $(V_n)_{n\geqslant 1}$ since it is finite over $\mathcal{W} \times \mathbb{A}^1$. We claim that the Zariski open subsets $V_n \cap \tilde{\mathcal{E}}^{\mathrm{irr}} \subset V_n$ can also be covered by an increasing union of affinoids $(V_{n,m})_{m\geqslant 1}$. Assuming the claim, we have $\tilde{\mathcal{E}}^{\mathrm{irr}} = \bigcup_{n,m} V_{n,m}$. By quasicompactness of affinoids we can find a sequence $m_1, m_2, \ldots$ such that V_{n,m_n} contains $V_{n-1,m_{n-1}}$ and $V_{i,j}$ for all $i \leqslant n$ and $j \leqslant n$, and therefore $\tilde{\mathcal{E}}^{\mathrm{irr}} = \bigcup_n V_{n,m_n}$ is covered by an increasing union of affinoids. To show the claim, it suffices to show that a Zariski open subset of a one-dimensional affinoid V is an increasing union of affinoids. If $U \subset V$ has complement cut out by functions $f_1, f_2, \ldots, f_k$ then $U = \bigcup_{n\geqslant 1} \bigcup_{i=1}^k \{|f_i| \geqslant 1/n\}$. The finite union of affinoids $\bigcup_{i=1}^k \{|f_i| \geqslant 1/n\} \subset V$ is itself affinoid, by [Fie80, Satz 2.1]. □

3.1.2. p-adic Hodge theoretic properties at $v' \neq v$. Now we let $v'|p$ be a place of F with $v' \neq v$.

PROPOSITION 3.1.8.

(1) *Let $z \in \tilde{\mathcal{E}}^{\mathrm{irr}}$. Then $V_z|_{G_{F_{v'}}}$ is potentially semistable with Hodge–Tate weights $-(w + k_{v'})/2$ and $-(1 + (w - k_{v'})/2)$.*

(2) *The restriction to $I_{F_{v'}}$ of the Weil–Deligne representation $WD(V_z|_{G_{F_{v'}}})$ depends only on the connected component of z in $\tilde{\mathcal{E}}^{\mathrm{irr}}$.*

(3) *If z is a classical point with $U_{v'}(z) = \alpha \neq 0$, and we set $\widetilde{\alpha} = \alpha p^{1+(w-k_{v'})/2}$ then the associated (Frobenius-semisimplified) Weil–Deligne representation is of the form*

$$WD(V_z|_{G_{F_{v'}}})^{F-ss} = \left(ur(\widetilde{\beta})^{-1}\psi_1 \oplus ur(\widetilde{\alpha})^{-1}\psi_2, N\right)$$

for some characters ψ_i of $\mathcal{O}_{F,v'}^\times$ and $\widetilde{\beta} \in \mathbb{C}_p$ (determined by the determinant of V_z) satisfying $v_p(\widetilde{\alpha}) + v_p(\widetilde{\beta}) = 1 + w$. N is nonzero if and only if $\psi_1 = \psi_2$ and $\widetilde{\alpha}/\widetilde{\beta} = p^{-1}$.

Proof. For classical points, the first and third assertions follow from local–global compatibility, as in Proposition 3.1.3. Note that for the third part, a calculation similar to Lemma 2.3.1 shows that we can assume that the integer n defining the level at the place v' is minimal such that the relevant space of invariants $\pi_{v'}^{I'_{v',1,n-1}}$ is nonzero. The rest of the Proposition then follows from [BC08, Theorems B, C], which were generalized to reduced quasicompact (and quasiseparated) rigid spaces in [Che, Corollary 3.19], and [BC09, Lemma 7.5.12] (which allows us to identify the Hodge–Tate weights precisely). Note that every point of $\tilde{\mathcal{E}}^{\mathrm{irr}}$ is contained in a quasicompact open subspace with a Zariski-dense set of classical points (as in the proof of Corollary 3.1.7, see [CHJ17, Lemma 5.9]). □

DEFINITION 3.1.9. We denote by $\tilde{\mathcal{E}}^{v'-\mathrm{fs}}$ the Zariski open subspace of $\tilde{\mathcal{E}}^{\mathrm{irr}}$ where $U_{v'}$ is nonzero.

LEMMA 3.1.10.

(1) *If $z \in \tilde{\mathcal{E}}^{v'-\mathrm{fs}}$, with $U_{v'}(z) = \alpha \neq 0$, and we set $\widetilde{\alpha} = \alpha p^{1+(w-k_{v'})/2}$ then the associated (Frobenius-semisimple) Weil–Deligne representation is of the form*

$$WD(V_z|_{G_{F_{v'}}})^{F-ss} = \left(ur(\widetilde{\beta})^{-1}\psi_1 \oplus ur(\widetilde{\alpha})^{-1}\psi_2, N\right)$$

for some characters ψ_i of $\mathcal{O}_{F,v'}^{\times}$ and $\widetilde{\beta} \in \mathbb{C}_p$ (determined by the determinant of V_z) satisfying $v_p(\widetilde{\alpha}) + v_p(\widetilde{\beta}) = 1 + w$.

(2) *$\tilde{\mathcal{E}}^{v'-\mathrm{fs}}$ is a union of connected components of $\tilde{\mathcal{E}}^{\mathrm{irr}}$.*

Proof. We begin by establishing the first part. Suppose $z \in \tilde{\mathcal{E}}^{v'-\mathrm{fs}}$. It follows from the second and third parts of Proposition 3.1.8 that the Weil–Deligne representation at z has a reducible Weil group representation. Since the Weil–Deligne representation varies analytically over $\tilde{\mathcal{E}}^{\mathrm{irr}}$ [**BC08**, Theorem C], we can consider the characteristic polynomial of a lift of $\mathrm{Frob}_{v'}$ over $\tilde{\mathcal{E}}^{\mathrm{irr}}$. The third part of Proposition 3.1.8 and Zariski density of classical points implies that $p^{1+(w-k_{v'})/2}U_{v'}$ is a root of this polynomial over $\tilde{\mathcal{E}}^{v'-\mathrm{fs}}$, and this proves that the Weil–Deligne representation has the desired form over all of $\tilde{\mathcal{E}}^{v'-\mathrm{fs}}$.

For the second part, we suppose that $z \in \tilde{\mathcal{E}}^{\mathrm{irr}}$ is an element of the Zariski closure of $\tilde{\mathcal{E}}^{v'-\mathrm{fs}}$. We need to show that $U_{v'}(z) \neq 0$. The same argument we used to establish the first part shows that the Weil–Deligne representation at z has a reducible Weil group representation with a $\mathrm{Frob}_{v'}$ eigenvalue given by $p^{1+(w-k_{v'})/2}U_{v'}(z)$. It follows immediately that $U_{v'}(z) \neq 0$. □

LEMMA 3.1.11. *The locus of points $z \in \tilde{\mathcal{E}}^{v'-\mathrm{fs}}$ with $V_z|_{G_{F_{v'}}}$ reducible is equal to the locus of points with $v_p(U_{v'}(z)) = 0$ or $k_{v'} - 1$.*

Moreover, this locus is a union of connected components of $\tilde{\mathcal{E}}^{v'-\mathrm{fs}}$.

Proof. It follows from Lemma 3.1.10 and the explicit description of admissible filtered Weil–Deligne modules [**Col08**, Section 4.5] that $V_z|_{G_{F_{v'}}}$ is reducible if and only if $v_p(U_{v'}(z)) = 0$ or $k_{v'} - 1$. Since reducibility of a representation is a Zariski closed condition, and the conditions on the slope are open, we deduce that the reducible locus is a union of connected components. □

DEFINITION 3.1.12. We denote the locus of $z \in \tilde{\mathcal{E}}^{v'-\mathrm{fs}}$ with $V_z|_{G_{F_{v'}}}$ irreducible or reducible with $v_p(U_{v'}(z)) = 0$ by $\tilde{\mathcal{E}}^{v'-\mathrm{fs,good}}$ (the preceding lemma implies that this is a union of connected components of $\tilde{\mathcal{E}}^{v'-\mathrm{fs}}$).

COROLLARY 3.1.13. *There are line bundles* $\mathscr{L}_1$ *and* $\mathscr{L}_2$ *over* $\tilde{\mathcal{E}}^{v'-\mathrm{fs,good}}$, *a continuous character* $\delta : F_{v'}^\times \to \Gamma(\tilde{\mathcal{E}}^{v'-\mathrm{fs,good}}, \mathcal{O}_{\tilde{\mathcal{E}}})^\times$ *with* $\delta(p) = U_{v'}^{-1}$ *and a short exact sequence*

$$0 \to \mathscr{R}_{\tilde{\mathcal{E}}^{v'-\mathrm{fs,good}}}(\delta^{-1}\det\rho)\otimes\mathscr{L}_1 \to \mathbf{D}_{\mathrm{rig}}(V_{\tilde{\mathcal{E}}^{v'-\mathrm{fs,good}}}|_{G_{F_{v'}}}) \to \mathscr{R}_{\tilde{\mathcal{E}}^{v'-\mathrm{fs,good}}}(\delta)\otimes\mathscr{L}_2 \to 0.$$

Proof. This is proved exactly as Corollary 3.1.7. □

Finally, we note that if $\delta : \mathbb{Q}_p^\times \to L^\times$ is a continuous character such that there exists $i \in \mathbb{Z}$ with $\delta(x) = x^i$ for all x in a finite index subgroup of $\mathbb{Z}_p^\times$, then $\mathscr{R}(\delta)$ is de Rham of Hodge–Tate weight $-i$ (as defined in [**PX**, Section 2.2]). We can now give the proof of Theorem 1.2.3.

Proof of Theorem 1.2.3. If $[F:\mathbb{Q}]$ is even, we let D/F be the totally definite quaternion algebra which is split at all finite places. If $[F:\mathbb{Q}]$ is odd, we let D/F be the totally definite quaternion algebra which is split at all finite places except for the fixed place v_0.

By the Jacquet–Langlands correspondence, we can find a compact open subgroup $K^p \subset (D\otimes_F \mathbb{A}^p_{F,f})^\times$, an integer $n \geqslant 1$, a finite order character $\epsilon = \prod_{v|p}\epsilon_v : \prod_{v|p} T_{0,v} \to E_\lambda^\times$ and a Hecke eigenform

$$f \in H^0\left(K^p\prod_{v|p} I_{v,1,n-1}, \bigotimes_{v|p}\mathcal{L}(k_v, 0, \epsilon_v)\right)$$

with associated Galois representation V_f isomorphic to $V_{g,\lambda}$. Moreover, if g is v-(nearly) ordinary at a place $v|p$ we choose f so that its U_v-eigenvalue α_v satisfies $v_p(\alpha_v) = 0$. In particular, we have $v_p(\alpha_v) < k_v - 1$ for all $v|p$.

Triviality of the central character of g implies that each character ϵ_v has trivial restriction to the diagonally embedded copy of $\mathcal{O}_v^\times$. The determinant $\det(V_f)$ is the cyclotomic character.

Now we choose an ordering $v_1, v_2, \ldots, v_d$ of the places $v|p$. We can now apply the results of Sections 2.7 and 3.1 fixing the place $v = v_1$. By Proposition 2.7.4, we deduce that we can find an integer $n_1 \geqslant n$, a finite extension $E_1/\mathbb{Q}_p$, a finite order character $\epsilon_1 = \prod_{v|p}\epsilon_{1,v} : \prod_{v|p} T_{0,v} \to E_1^\times$ and a Hecke eigenform $f_1 \in H^0(K^p\prod_{v|p} I_{v,1,n_1-1}, \mathcal{L}(2,0,\epsilon_{1,v_1})\bigotimes_{i=2}^d \mathcal{L}(k_{v_i}, 0, \epsilon_{1,v_i}))$ such that f and f_1 give rise to classical points of a common irreducible component C of $\mathcal{E}(k^{v_1}, 0)$.

Moreover, the U_v-eigenvalues α_v of f_1 satisfy $v(\alpha_v) < k_v - 1$ for all $v|p$. For v_1 this is part of the statement of Proposition 2.7.4. For the other places, this follows from Lemma 3.1.11.

Now we can apply Corollaries 3.1.13 and 3.1.7, together with [**PX**, Theorem A] to deduce that the validity of the parity conjecture for $V_{g,\lambda}$ is equivalent to the validity of the parity conjecture for V_{f_1}. For the reader's convenience, we reproduce here the statement of [**PX**, Theorem A]:

THEOREM. *Let X be an irreducible reduced rigid analytic space over $\mathbb{Q}_p$, and $\mathscr{T}$ a locally free coherent $\mathcal{O}_X$-module equipped with a continuous, $\mathcal{O}_X$-linear action of $G_{F,S}$, and a skew-symmetric isomorphism $j : \mathscr{T} \xrightarrow{\sim} \mathscr{T}^*(1)$. Assume given, for each $v_i|p$, a short exact sequence*

$$\mathscr{S}_i : 0 \to \mathscr{D}_i^+ \to \mathbf{D}_{\mathrm{rig}}(\mathscr{T}|_{G_{F_{v_i}}}) \to \mathscr{D}_i^- \to 0$$

of (φ, Γ)-modules over $\mathscr{R}_X$, with $\mathscr{D}_i^+$ Lagrangian with respect to j.

*For a (closed) point $P \in X$, we put $V = \mathscr{T} \otimes_{\mathcal{O}_X} \kappa(P)$ and $S_i = \mathscr{S}_i \otimes_{\mathcal{O}_X} \kappa(P)$. Let X_{alg} be the set of points $P \in X$ such that (1) for all i, the sequence S_i describes $\mathbf{D}_{\mathrm{rig}}(V|_{G_{F_{v_i}}})$ as an extension of a de Rham (φ, Γ)-module with nonnegative Hodge–Tate weights by a de Rham (φ, Γ)-module with negative Hodge–Tate weights (this is called the 'Panchiskin condition' in [**PX**]), and (2) for all $w \in S\backslash S_\infty$ where $\mathscr{T}|_{G_{F_w}}$ is ramified, the associated Weil–Deligne representation $WD(V|_{G_{F_w}})$ is pure.*

Then the validity of the parity conjecture for V, relating the sign of its global ε-factor to the parity of the dimension of its Bloch–Kato Selmer group, is independent of $P \in X_{\mathrm{alg}}$.

Letting $U \subset \tilde{\mathscr{E}}$ be a Zariski open subset containing every noncritical classical point, as in the second part of Corollary 3.1.7, we apply the above theorem with $X = U \cap (\bigcap_{v' \neq v_1 | p} \tilde{\mathscr{E}}^{v'-\mathrm{fs,good}})$ (note that X contains the points arising from both f and f_1). The short exact sequences $\mathscr{S}_i$ are given by Corollary 3.1.7 (for $i = 1$) and Corollary 3.1.13 (for the other i). We can fix a skew-symmetric isomorphism j (it exists since our Galois representations have cyclotomic determinant). The specialization of $\mathscr{D}_i^+$ at a noncritical classical point of weight $(k_1, 0)$ is de Rham of Hodge–Tate weight $-k_i/2$, and the specialization of $\mathscr{D}_i^-$ is de Rham of Hodge–Tate weight $(k_i - 2)/2$ (see Proposition 3.1.6). It follows from this and local–global compatibility at places in $S\backslash S_\infty$ that X_{alg} contains all the noncritical classical points of X (in particular, the points arising from f and f_1). Therefore the validity of the parity conjecture for V_f is equivalent to the validity of the parity conjecture for V_{f_1}.

We apply this argument $d-1$ more times, and deduce that the validity of the parity conjecture for $V_{g,\lambda}$ is equivalent to the validity of the parity conjecture for V_{f_d}, for a Hecke eigenform

$$f_d \in H^0\left(K^p \prod_{v|p} I_{v,1,n_d-1}, \bigotimes_{i=1}^{d} \mathcal{L}(2,0,\epsilon_{d,v_i})\right).$$

We moreover know that f_d does not generate a one-dimensional representation of $(D \otimes_F \mathbb{A}_F)^\times$ (since it has noncritical slope). Finally, by the Jacquet–Langlands correspondence and [Nek18, Theorem C], the parity conjecture holds for V_{f_d} (note that our Galois representations all have cyclotomic determinant, so the automorphic representation associated to f_d has trivial central character), so the parity conjecture holds for V_f. □

4. The full eigenvariety

4.1. Notation and preliminaries. In this section we will consider an eigenvariety for automorphic forms on the quaternion algebra D where the weights are allowed to vary at all places $v|p$. The set-up is very similar to Section 2.5 so we will be brief.

We set $T_0 = \prod_{v|p} T_{0,v}$, $I_1 = \prod_{v|p} I_{v,1}$ and $I_{1,n} = \prod_{v|p} I_{v,1,n}$, and fix a compact open subgroup $K^p = \prod_{v\nmid p} K_v \subset (\mathcal{O}_D \otimes_F \mathbb{A}^p_{F,f})^\times$ such that $K = K^p I_1$ is neat.

We let X_p denote the kernel of $N_{F/\mathbb{Q}} : \mathcal{O}^\times_{F,p} \to \mathbb{Z}_p^\times$.

The weight space $\mathcal{W}^{\text{full}}$ is defined by letting

$$\mathcal{W}^{\text{full}}(A) = \{\kappa \in \operatorname{Hom}_{cts}(T_0, A^\times) : \kappa|_{Z(K)} = 1 \text{ and } \kappa|_{X_p} \text{ has finite order}\}$$

for $\mathbb{Q}_p$-affinoid algebras A. We have $\dim(\mathcal{W}^{\text{full}}) = 1+d$ and the locally algebraic points (defined as in the partial case) are Zariski-dense in $\mathcal{W}^{\text{full}}$. If the p-adic Leopoldt conjecture holds for F then $Z(K)$ has closure in X_p a finite index subgroup, and the second condition in the definition of the points of weight space is automatic.

For weights $\kappa \in \mathcal{W}^{\text{full}}(A)$ (together with a norm on A which is adapted to κ) and $r \geqslant r_\kappa$ we get a Banach A-module of distributions $\mathcal{D}^r_\kappa$ equipped with a left action of the monoid $\prod_{v|p} \Delta_v$. We therefore obtain an A-module $H^0(K, \mathcal{D}^r_\kappa)$ equipped with an action of the Hecke algebra $\mathbb{T}$.

We let $U_p = \prod_{v|p} U_v$, which is a compact operator on $H^0(K, \mathcal{D}^r_\kappa)$. We have its spectral variety $\mathcal{Z}^{\text{full}} \to \mathcal{W}^{\text{full}}$ and a coherent sheaf $\mathcal{H}^{\text{full}}$ on $\mathcal{Z}^{\text{full}}$ coming from the modules $H^0(K, \mathcal{D}^r_\kappa)$, equipped with a map $\psi : \mathbb{T} \to \operatorname{End}(\mathcal{H}^{\text{full}})$. We write $\mathcal{E}^{\text{full}}$ for the eigenvariety associated to the eigenvariety datum $(\mathcal{W}^{\text{full}} \times \mathbb{A}^1, \mathcal{H}^{\text{full}}, \mathbb{T}, \psi)$ (or, equivalently, $(\mathcal{Z}^{\text{full}}, \mathcal{H}^{\text{full}}, \mathbb{T}, \psi)$). $\mathcal{E}^{\text{full}}$ is reduced and equidimensional of dimension $1+d$, with a Zariski-dense subset of classical points which we now describe.

Given integers $k=(k_v)_{v|p}$ and w all of the same parity, and a finite order character $\epsilon : T_0 \to E^\times$ which is trivial on $\prod_{v|p}(1+\varpi_v^n\mathcal{O}_{F_v})^2$, we have an $I_{1,n-1}$-representation $\mathcal{L}(k,w,\epsilon)=\otimes_{v|p}\mathcal{L}_v(k_v,w,\epsilon)$. As in Section 2.2, k, w and ϵ correspond to a character $\kappa : T_0 \to E^\times$, which we suppose is a point of $\mathcal{W}^{\mathrm{full}}$ (in other words we suppose $\kappa|_{Z(K)}=1$ and $\kappa|_{X_p}$ has finite order). We say that κ is locally algebraic of weight (k,w) and character ϵ.

We then obtain a map (as in (2.3.1))

$$\pi : H^0(K,\mathcal{D}_\kappa^r) \to H^0(K^pI_{1,n-1},\mathcal{L}(k,w,\epsilon)),$$

where for each $v|p$ the action of U_v on the target is the $\star$-*action* defined by multiplying the standard action of U_v by $p^{-(w-k_v+2)/2}$.

PROPOSITION 4.1.1. *For each $v|p$, let $h_v\in\mathbb{Q}_{\geqslant 0}$ with $h_v<k_v-1$. The map π induces an isomorphism between the subspaces where U_v acts with slope $\leqslant h_v$ for all $v|p$.*

Proof. As for Proposition 2.3.3, this can be proved using the method of [**Han17**, Theorem 3.2.5]. □

We say that a point of $\mathcal{E}^{\mathrm{full}}$ is classical if its weight is locally algebraic and the point corresponds to a Hecke eigenvector with nonzero image under the map π above. We say that a classical point (with weight (k,w)) has noncritical slope if the corresponding U_v-eigenvalues have slope $<k_v-1$ for each $v|p$.

LEMMA 4.1.2. *Let $z\in\mathcal{E}^{\mathrm{full}}$ be a classical point of weight (k,w) and character ϵ with noncritical slope. Let $\epsilon=(\epsilon_1,\epsilon_2)$, where ϵ_i is a character of $\mathcal{O}_{F,p}^\times$. If $\epsilon_1|_{\mathcal{O}_{F_v}^\times}=\epsilon_2|_{\mathcal{O}_{F_v}^\times}$ suppose moreover that $v_p(U_v(z))\neq(k_v-1)/2$. Then $\mathcal{E}^{\mathrm{full}}$ is étale over $\mathcal{W}^{\mathrm{full}}$ at z.*

Proof. This can be proved as in [**Che11**, Theorems 4.8, 4.10]. Under our assumptions, the Hecke algebra $\mathbb{T}$ acts semisimply on the space of classical automorphic forms of fixed weight and character. We denote the residue field $k(\kappa(z))$ by L. Replacing z by a point (which we also call z) lying over it in $\mathcal{E}_L^{\mathrm{full}}$, it suffices to show that $\mathcal{E}_L^{\mathrm{full}}$ is étale over $\mathcal{W}_L^{\mathrm{full}}$ at z. By the construction of the eigenvariety, we can suppose that we have a geometrically connected (smooth) L-affinoid neighbourhood B of the weight $\kappa(z)\in\mathcal{W}_L^{\mathrm{full}}$ and a finite locally free $\mathcal{O}(B)$-module M, equipped with an $\mathcal{O}(B)$-linear $\mathbb{T}$-action such that:

 https://doi.org/10.1017/fms.2019.23

(1) The affinoid spectrum V of the image of $\mathcal{O}(B) \otimes \mathbb{T}$ in $\mathrm{End}_{\mathcal{O}(B)}(M)$ is an open neighbourhood of $z \in \mathcal{E}^{\mathrm{full}}$.

(2) For each point $x \in B$ and algebraic closure $\overline{k(x)}$ of the residue field $k(x)$ there is an isomorphism of $\overline{k(x)} \otimes \mathbb{T}$-modules

$$M \otimes_{\mathcal{O}(B)} \overline{k(x)} \cong \bigoplus_{y \in \kappa^{-1}(x) \cap V} \bigoplus_{\iota \in \mathrm{Hom}_{k(x)}(k(y), \overline{k(x)})} H^0(K, \mathcal{D}_x^r) \otimes_{k(x)} \overline{k(x)}[\iota \circ \psi_y],$$

where ψ_y denotes the ($k(y)$-valued) system of Hecke eigenvalues associated to y, and $[\iota \circ \psi_y]$ denotes the generalized eigenspace.

(3) $\kappa^{-1}(\kappa(z)) = \{z\}$ and the natural L-algebra surjection $\mathcal{O}(V) \to k(z)$ has a section.

Now we apply the classicality criterion (Proposition 4.1.1), strong multiplicity one, and [**Che11**, Lemma 4.7] as in [**Che11**, Theorem 4.10] to deduce that there is a Zariski-dense subset $Z_0 \subset B$ of locally algebraic weights with residue field L such that κ is étale at each point of $\kappa^{-1}(Z_0)$, and for each $\chi \in Z_0$, $\kappa^{-1}(\chi) \cap V$ consists of a single point with residue field $k(z)$.

We can now show that the map $k(z) \otimes_L \mathcal{O}(B) \to \mathcal{O}(V)$ obtained from assumption (3) is an isomorphism. Since $k(z) \otimes_L \mathcal{O}(B)$ is a normal integral domain, it suffices to show that the map $\pi : \mathrm{Spec}(\mathcal{O}(V)) \to \mathrm{Spec}(k(z) \otimes_L \mathcal{O}(B))$ is birational. By generic flatness, there is a dense open subscheme U of the target such that $\pi|_U$ is finite flat. Since π maps irreducible components surjectively onto irreducible components, $\pi^{-1}(U)$ is a dense open subscheme of the source. Since $Z_0 \cap U$ is nonempty, $\pi|_U$ has degree one and is therefore an isomorphism, which shows that π is birational as desired. In fact, it is not necessary to use the normality of B (as explained to us by Chenevier)—$\mathcal{O}(V)$ is a subalgebra of $\mathrm{End}_{k(z) \otimes_L \mathcal{O}(B)}(k(z) \otimes_L M)$ and each element of $\mathcal{O}(V)$ acts as a scalar in $k(z)$ when we specialize at a weight $\chi \in Z_0$. Since B is reduced and Z_0 is Zariski-dense in B, this is enough to conclude that each element of $\mathcal{O}(V)$ acts as a scalar (in $k(z) \otimes_L \mathcal{O}(B)$) on $k(z) \otimes_L M$. □

4.2. Mapping from partial eigenvarieties to the full eigenvariety. Now we fix a place $v|p$ and consider the eigenvariety $\mathcal{E}(k^v, w)$ for fixed k^v, w as in Section 2.7. For this partial eigenvariety, we fix the level structure $K'_{v'} = I'_{v',1,n-1}$ for each place $v'|p$ with $v' \neq v$, and let $K' = K^p I_{v,1} \prod_{v' \neq v} K'_{v'}$. We denote by $\mathcal{E}(k^v, w)^{U_p - fs} \subset \mathcal{E}(k^v, w)$ the union of irreducible components (with the reduced subspace structure) given by the image of $\bigcap_{v \neq v'|p} \widetilde{\mathcal{E}}^{v'-fs}$ (see Definition 3.1.9).

We can also construct a spectral variety $\mathcal{Z}^{U_p}(k^v, w)$ and eigenvariety $\mathcal{E}^{U_p}(k^v, w)$ using the compact operator U_p instead of U_v. The closed points of $\mathcal{E}^{U_p}(k^v, w)$ and $\mathcal{E}(k^v, w)^{U_p-fs}$ correspond to the same systems of Hecke eigenvalues, so the following lemma should be no surprise.

LEMMA 4.2.1. *There is a canonical isomorphism $\mathcal{E}^{U_p}(k^v, w)^{\mathrm{red}} \cong \mathcal{E}(k^v, w)^{U_p-fs}$, over $\mathcal{W}$, compatible with the Hecke operators.*

Proof. Suppose we have a slope datum (U, h) (see [JN19b, Definition 2.3.1]) corresponding to an open subset $\mathcal{Z}_{U,h}^{U_p} \subset \mathcal{Z}^{U_p}(k^v, w)$. We therefore have a slope decomposition

$$H^0(K', \mathcal{L}^v(k^v, w) \otimes_{\mathbb{Z}_p} \mathcal{D}_U^r) = M = M_{\leqslant h} \oplus M_{>h}$$

and a corresponding factorization $\det(1 - U_v T) = QS$ of the characteristic power series of U_v on M, so we obtain a closed immersion

$$Z(Q) := \{Q = 0\} \hookrightarrow \mathcal{Z}^{U_v}(k^v, w)|_U.$$

The module $M_{\leqslant h}$ defines a coherent sheaf on $Z(Q)$.

Gluing, we obtain a rigid space $\mathcal{Z}^{U_p,U_v}$ equipped with a closed immersion $\mathcal{Z}^{U_p,U_v} \hookrightarrow \mathcal{Z}^{U_v}(k^v, w)$, together with a coherent sheaf $\mathcal{H}^{U_p,U_v}$ on $\mathcal{Z}^{U_p,U_v}$.

Over $\mathcal{Z}_{U,h}^{U_p}$, $\mathcal{E}^{U_p}(k^v, w)$ is given by the spectrum of the affinoid algebra

$$\mathcal{T}_{U,h} = \mathrm{im}(\mathbb{T} \otimes_{\mathbb{Z}_p} \mathcal{O}(U) \to \mathrm{End}_{\mathcal{O}(U)} M_{\leqslant h}).$$

The eigenvariety associated to the datum $(\mathcal{W} \times \mathbb{A}^1, \mathcal{H}^{U_p,U_v}, \mathbb{T}, \psi)$ has the same description (since $U_v \in \mathbb{T}$), so these two eigenvarieties are canonically isomorphic.

On the other hand, we can identify the closed points of $\mathcal{E}(k^v, w)^{U_p-fs}$ and $\mathcal{E}^{U_p}(k^v, w)$, since they correspond to the same systems of Hecke eigenvalues. It follows from [JN19a, Theorem 3.2.1], applied to the eigenvariety data $(\mathcal{W} \times \mathbb{A}^1, \mathcal{H}^{U_p,U_v}, \mathbb{T}, \psi)$ and $(\mathcal{W} \times \mathbb{A}^1, \mathcal{H}, \mathbb{T}, \psi)$ that there is a canonical closed immersion $\mathcal{E}^{U_p}(k^v, w)^{\mathrm{red}} \hookrightarrow \mathcal{E}(k^v, w)$, which induces the desired isomorphism

$$\mathcal{E}^{U_p}(k^v, w)^{\mathrm{red}} \cong \mathcal{E}(k^v, w)^{U_p-fs}. \qquad \square$$

To compare $\mathcal{E}(k^v, w)^{U_p-fs}$ with $\mathcal{E}^{\mathrm{full}}$, we need to fix a character

$$\epsilon^v : \prod_{v' \neq v} I_{v',1,n-1}/I'_{v',1,n-1} \to \mathbb{Q}_p^\times,$$

and consider the eigenvarieties $\mathcal{E}^{U_p}(k^v, w, \epsilon^v)$, $\mathcal{E}(k^v, w, \epsilon^v)$ given by replacing the coherent sheaf $\mathcal{H}$ in the eigenvariety data with the ϵ^v-isotypic piece $\mathcal{H}(\epsilon^v)$.

These are unions of connected components in the original eigenvarieties. They are supported over the union of connected components $\mathcal{W}(\epsilon^v) \subset \mathcal{W}$ given by weights κ_v such that $\kappa_v \epsilon^v \chi^v|_{Z(K)} = 1$, where χ^v is the highest weight of $\mathcal{L}^v(k^v, w)$. Note that in general points $\kappa_v \in \mathcal{W}$ satisfy the weaker condition that $\kappa_v \chi^v|_{Z(K')} = 1$.

DEFINITION 4.2.2. We write $\iota_{(k^v,w,\epsilon^v)} : \mathcal{W}(\epsilon^v) \hookrightarrow \mathcal{W}^{\mathrm{full}}$ for the closed immersion defined by

$$\iota_{(k^v,w,\epsilon^v)}(\kappa_v) = \kappa_v \epsilon^v \chi^v,$$

where χ^v is the highest weight of $\mathcal{L}^v(k^v, w)$.

LEMMA 4.2.3. *There is a canonical closed immersion*

$$\mathcal{E}(k^v, w, \epsilon^v)^{U_p-fs} \hookrightarrow \mathcal{E}^{\mathrm{full}},$$

lying over the map $\iota_{(k^v,w,\epsilon^v)} : \mathcal{W}(\epsilon^v) \hookrightarrow \mathcal{W}^{\mathrm{full}}$, *compatible with the Hecke operators.*

Proof. Applying Lemma 4.2.1, we replace $\mathcal{E}(k^v, w, \epsilon^v)^{U_p-fs}$ in the statement with $\mathcal{E}^{U_p}(k^v, w, \epsilon^v)^{\mathrm{red}}$. The classical points of $\mathcal{E}^{U_p}(k^v, w, \epsilon^v)^{\mathrm{red}}$ naturally correspond to a subset of the classical points of $\mathcal{E}^{\mathrm{full}}$, and we apply [**JN19a**, Theorem 3.2.1] to obtain the desired closed immersion. □

COROLLARY 4.2.4. *Let C be an irreducible component of the full eigenvariety $\mathcal{E}^{\mathrm{full}}$. Then C contains a noncritical slope classical point of weight* $(2, \dots, 2, 0)$ *(or any other algebraic weight).*

Proof. First we note that C contains a noncritical slope classical point x_0 of weight $(k_1, \dots, k_d, w)$ and character $\epsilon = (\epsilon_1, \epsilon_2)$ with w even (since the image of C in weight space is Zariski open). We can moreover assume that the characters $\epsilon_1|_{\mathcal{O}_{F_v}^\times}$ and $\epsilon_2|_{\mathcal{O}_{F_v}^\times}$ are distinct for all $v|p$ (for example by considering weights in a sufficiently small 'boundary poly-annulus' of the weight space). We do this to ensure that x_0 will be étale over weight space. As in the proof of Theorem 1.2.3, we obtain a sequence $x_1, x_2, \dots, x_d$ of noncritical slope classical points where each x_i has a weight $k(i)$ with $k(i)_1 = k(i)_2 = \cdots = k(i)_i = 2$, with the remaining weight components the same as x_0. The v_{i+1} part of the character ϵ changes when we move from x_i to x_{i+1}, but it keeps the property that it has distinct components. So, by Lemma 4.1.2, the map from $\mathcal{E}^{\mathrm{full}}$ to $\mathcal{W}^{\mathrm{full}}$ is étale at each of the points x_i.

For each i, x_i and x_{i+1} lie in the image of an irreducible component of $\mathcal{E}(k(i)^{v_{i+1}}, w, \epsilon^{v_{i+1}})^{U_p-fs}$ under the closed immersion of Lemma 4.2.3 (for some choice of $\epsilon^{v_{i+1}}$). It follows that x_i and x_{i+1} lie in a common irreducible component of $\mathcal{E}^{\mathrm{full}}$.

 https://doi.org/10.1017/fms.2019.23

Since the map from $\mathcal{E}^{\text{full}}$ to $\mathcal{W}^{\text{full}}$ is étale at each of the points x_i, x_i is a smooth point of $\mathcal{E}^{\text{full}}$ and is therefore contained in a unique irreducible component of $\mathcal{E}^{\text{full}}$. We deduce that $x_i \in C$ for all i. The point x_d has weight $(2, \ldots, 2, w)$. Finally, there is a one-dimensional family of twists by powers of the cyclotomic character connecting x_d to a noncritical slope classical point of weight $(2, \ldots, 2, 0)$, which gives the desired point of C. □

REMARK 4.2.5. At least for irreducible components C whose associated mod p Galois representation $\overline{\rho}$ is absolutely irreducible, one can deduce cases of the parity conjecture directly from the above corollary, using [PX, Theorem A]. Without this assumption on $\overline{\rho}$, the Galois pseudocharacter over C cannot be automatically lifted to a genuine family of Galois representations over C. This is why our proof of Theorem 1.2.3 instead uses one-dimensional families coming from the partial eigenvarieties, where we can apply Proposition 3.1.1.

Acknowledgements

The authors wish to thank Liang Xiao for useful discussions on the paper [PX], together with John Bergdall, Gaëtan Chenevier and Jan Nekovář for helpful correspondence. The authors also thank an anonymous referee for helpful comments. C.J. was supported by NSF grant DMS 1128155 and by the Herchel Smith Foundation during various stages of this work.

References

[BDJ17] D. Barrera, M. Dimitrov and A. Jorza, ‘p-adic L-functions of Hilbert cusp forms and the trivial zero conjecture’. ArXiv e-prints, September 2017.

[BC09] J. Bellaïche and G. Chenevier, ‘Families of Galois representations and Selmer groups’, *Astérisque* **324** (2009), xii+314.

[Ber08] L. Berger, ‘Équations différentielles p-adiques et (ϕ, N)-modules filtrés’, *Astérisque* **319** (2008), 13–38. Représentations p-adiques de groupes p-adiques. I. Représentations galoisiennes et (ϕ, Γ)-modules.

[BC08] L. Berger and P. Colmez, ‘Familles de représentations de de Rham et monodromie p-adique’, *Astérisque* **319** (2008), 303–337. Représentations p-adiques de groupes p-adiques. I. Représentations galoisiennes et (ϕ, Γ)-modules.

[Buz07] K. Buzzard, ‘Eigenvarieties’, in *L-functions and Galois Representations*, London Mathematical Society Lecture Note Series, 320 (Cambridge University Press, Cambridge, 2007), 59–120.

[Che] G. Chenevier, ‘Une application des variétés de Hecke des groupes unitaires’, in *Shimura Varieties*, London Mathematical Society Lecture Note Series (Cambridge University Press, Cambridge, in press).

[Che04] G. Chenevier, ‘Familles p-adiques de formes automorphes pour GL_n’, *J. Reine Angew. Math.* **570** (2004), 143–217.

[Che11] G. Chenevier, 'On the infinite fern of Galois representations of unitary type', *Ann. Sci. Éc. Norm. Supér. (4)* **44**(6) (2011), 963–1019.

[Che14] G. Chenevier, 'The p-adic analytic space of pseudocharacters of a profinite group and pseudorepresentations over arbitrary rings', in *Automorphic Forms and Galois Representations,* Vol. 1, London Mathematical Society Lecture Note Series, 414 (Cambridge University Press, Cambridge, 2014), 221–285.

[CH13] G. Chenevier and M. Harris, 'Construction of automorphic Galois representations, II', *Camb. J. Math.* **1**(1) (2013), 53–73.

[CHJ17] P. Chojecki, D. Hansen and C. Johansson, 'Overconvergent modular forms and perfectoid Shimura curves', *Doc. Math.* **22** (2017), 191–262.

[CM98] R. Coleman and B. Mazur, 'The eigencurve', in *Galois Representations in Arithmetic Algebraic Geometry (Durham, 1996)*, London Mathematical Society Lecture Note Series, 254 (Cambridge University Press, Cambridge, 1998), 1–113.

[Col08] P. Colmez, 'Représentations triangulines de dimension 2', *Astérisque* **319** (2008), 213–258. Représentations p-adiques de groupes p-adiques. I. Représentations galoisiennes et (ϕ, Γ)-modules.

[Din18] Y. Ding, 'Some results on the locally analytic socle for $\mathrm{GL}_n(\mathbb{Q}_p)$', *Int. Math. Res. Not. IMRN* (2018), rnx287.

[Fie80] K.-H. Fieseler, 'Zariski's main theorem für affinoide Kurven', *Math. Ann.* **251**(2) (1980), 97–110.

[Ger19] D. Geraghty, 'Modularity lifting theorems for ordinary Galois representations', *Math. Ann.* **373**(3–4) (2019), 1341–1427.

[Han17] D. Hansen, 'Universal eigenvarieties, trianguline Galois representations, and p-adic Langlands functoriality', *J. Reine Angew. Math.* **730** (2017), 1–64.

[JN19a] C. Johansson and J. Newton, 'Irreducible components of extended eigenvarieties and interpolating Langlands functoriality', *Math. Res. Lett.* **26**(1) (2019), 159–201.

[JN19b] C. Johansson and J. Newton, 'Extended eigenvarieties for overconvergent cohomology', *Algebra Number Theory* **13**(1) (2019), 93–158.

[dJvdP96] J. de Jong and M. van der Put, 'Étale cohomology of rigid analytic spaces', *Doc. Math.* **1**(01) (1996), 1–56.

[KPX14] K. S. Kedlaya, J. Pottharst and L. Xiao, 'Cohomology of arithmetic families of (φ, Γ)-modules', *J. Amer. Math. Soc.* **27**(4) (2014), 1043–1115.

[Liu12] T. Liu, 'Lattices in filtered (ϕ, N)-modules', *J. Inst. Math. Jussieu* **11**(3) (2012), 659–693.

[LWX17] R. Liu, D. Wan and L. Xiao, 'The eigencurve over the boundary of weight space', *Duke Math. J.* **166**(9) (2017), 1739–1787.

[Loe11] D. Loeffler, 'Overconvergent algebraic automorphic forms', *Proc. Lond. Math. Soc. (3)* **102**(2) (2011), 193–228.

[Mil80] J. S. Milne, *Étale Cohomology*, Princeton Mathematical Series, 33 (Princeton University Press, Princeton, NJ, 1980).

[Nek06] J. Nekovář, 'Selmer complexes', *Astérisque* **310** (2006), viii+559.

[Nek07] J. Nekovář, 'On the parity of ranks of Selmer groups. III', *Doc. Math.* **12** (2007), 243–274.

[Nek13] J. Nekovář, 'Some consequences of a formula of Mazur and Rubin for arithmetic local constants', *Algebra Number Theory* **7**(5) (2013), 1101–1120.

[Nek18] J. Nekovář, 'Compatibility of arithmetic and algebraic local constants, II: the tame abelian potentially Barsotti–Tate case', *Proc. Lond. Math. Soc. (3)* **116**(2) (2018), 378–427.

[PX] J. Pottharst and L. Xiao, 'On the parity conjecture in finite-slope families', Preprint, http://arxiv.org/abs/1410.5050.

[vdP82] M. van der Put, 'Cohomology on affinoid spaces', *Compositio Math.* **45**(2) (1982), 165–198.

[Sai09] T. Saito, 'Hilbert modular forms and p-adic Hodge theory', *Compositio Math.* **145**(5) (2009), 1081–1113.

[Ser98] J.-P. Serre, *Abelian l-adic Representations and Elliptic Curves*, Research Notes in Mathematics, 7 (A K Peters, Ltd, Wellesley, MA, 1998); With the collaboration of Willem Kuyk and John Labute, Revised reprint of the 1968 original.

[Ski09] C. Skinner, 'A note on the p-adic Galois representations attached to Hilbert modular forms', *Doc. Math.* **14** (2009), 241–258.

[Tem14] N. Templier, 'Large values of modular forms', *Camb. J. Math.* **2**(1) (2014), 91–116.